RESEARCH PROJECT

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My research works and projects lie at the intersection between geometric structures and dynamical systems.

In the one hand, I am interested with rigidity phenomena for differentiable dynamical systems, especially of hyperbolic or partially hyperbolic type, and specifically when their invariant distributions are highly regular. In this setting, I obtained in [MM22] a classification result for three-dimensional partially hyperbolic diffeomorphisms of contact type with smooth invariant distributions, presented in Theorem A below. This result is obtained by studying an invariant rigid geometric structure called *path geometry*, with the tools of *Cartan geometries*. Both of these notions are introduced in paragraph 1.3, together with a related rigidity result obtained in collaboration with Elisha Falbel and Jose Miguel Veloso in [FMMV21], see Theorem B. Following these two results, my first broad research project is to pursue a systematic study of the rigidity of partially hyperbolic diffeomorphisms having smooth invariant distributions, by the means of rigid geometric structures and Cartan geometries. I will present in paragraphs 2.1 and 2.2 two ongoing projects in this direction.

I am also interested with the dual problem, aiming at describing those compact rigid geometric structures that have a non-compact automorphism group. The research of new such examples in the case of *flat path geometries* led me to construct in [MM21] a geometric compactification of the geodesic flow of complete and non-compact hyperbolic surfaces, see Theorem C. These are examples of closed three-manifolds locally modelled on the *flag space* $\mathrm{PGL}_3(\mathbb{R})/\mathbf{P}_{min}$ (with $\mathbf{P}_{min}$ the Borel subgroup of upper triangular matrices), a geometry that is not fully understood yet and generally interests me. In this second broad direction of research, I will present in paragraph 2.3 a specific question that I am interested with.

The third aspect of my current research is concerned with three-dimensional flows. I am interested with *Anosov flows* and their $\mathbb{R}$-*covered property* in a project in collaboration with Federico Salmoiraghi and Mario Shannon that I will present in paragraph 2.4. In a second project in collaboration with Tali Pinsky that I will present in paragraph 2.5, we study a topological characterization of *conservative flows* up to orbital equivalence.

1. Contributions

1.1. **Contact-Anosov flows.** Let us recall that a non-singular flow (φ^t) of class $\mathcal{C}^\infty$ of a closed manifold M is called *Anosov*, if its differential preserves a splitting $\mathrm{T}M = E^s \oplus E^c \oplus E^u$ of the tangent bundle, where E^c is the direction of the flow and E^s and E^u are non-trivial distributions verifying the following estimates (with respect to any Riemannian metric on M).

(1) The *stable distribution* E^s is *uniformly contracted* by (φ^t), *i.e.* there are two constants $C>0$ and $0<\lambda<1$ such that for any $t\in\mathbb{R}$ and $x\in M$:

$$\left\|\mathrm{D}_x\varphi^t|_{E^s}\right\| \leq C\lambda^t. \tag{1.1}$$

(2) The *unstable distribution* E^s is *uniformly expanded* by (φ^t), *i.e.* uniformly contracted by (φ^{-t}).

Important examples of three-dimensional Anosov flows are given by the geodesic flows of closed hyperbolic surfaces Σ, acting on their unitary tangent bundle $\mathrm{T}^1\Sigma$. These flows have the following specific properties among Anosov flows: their stable and unstable distributions are $\mathcal{C}^\infty$ (while they are in general only Hölder continuous), and the sum $E^s\oplus E^u$ is furthermore a *contact distribution*. We recall that a plane field ξ of a three-dimensional manifold is called *contact* if it is nowhere integrable, or more precisely if it is locally the kernel of a *contact form* θ, *i.e.* a one-forme such

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that $\theta \wedge d\theta$ does not vanish. A beautiful result of Étienne Ghys in [Ghy87] says that, up to finite coverings and orbit equivalence[1], the geodesic flows of closed hyperbolic surfaces are in fact the only examples of three-dimensional Anosov flows whose stable and unstable distributions are $\mathcal{C}^\infty$ and such that $E^s \oplus E^u$ is a contact distribution. Ghys actually proves that all these flows are smoothly conjugated to algebraic examples (the right diagonal flow on compact quotients of $\mathrm{PSL}_2(\mathbb{R})$), and we will thus call them the *algebraic contact-Anosov flows.*

1.2. **Partially hyperbolic diffeomorphisms of contact type.** The result of [Ghy87] is a striking expression of the dynamical rigidity that can be deduced from geometrical assumptions for the case of flows, *i.e.* *continuous-time* dynamical systems. A thrilling question is then to know if these results generalize to *discrete-time* dynamical systems. Natural discrete-time analogs for the Anosov flows are the diffeomorphisms f of closed manifolds M, whose differential preserves a splitting $\mathrm{T}M = E^s \oplus E^c \oplus E^u$ (within non-zero distributions) such that E^s (respectively E^u) is uniformly contracted (resp. expanded) by $\mathrm{D}f$. These diffeomorphisms are called *partially hyperbolic*[2] (see [CP15] for a comprehensive introduction, and [HP18] for a general survey about the classification problem) and received a lot of attention in the last decades. In this setting, I obtained the following result.

Theorem A ([MM22, Theorem A])**.** *Let f be a partially hyperbolic diffeomorphism of a three-dimensional connected compact manifold M, whose invariant distributions E^s, E^u and E^c are smooth, such that $E^s \oplus E^u$ is a contact distribution, and whose non-wandering set $NW(f)$ equals M. Then, up to finite coverings and iterates, f is $\mathcal{C}^\infty$-conjugated to one of the following examples:*

(1) *the time-one map of a three-dimensional algebraic contact-Anosov flow;*
(2) *or a partially hyperbolic affine automorphism of a nil-*$\mathrm{Heis}(3)$*-manifold.*

Note that any diffeomorphism preserving a volume form satisfies the assumption $NW(f) = M$. The second family of examples are defined on compact quotients $\Gamma\backslash\mathrm{Heis}(3)$ of the three-dimensional Heisenberg group by cocompact lattices, and induced by affine automorphisms of $\mathrm{Heis}(3)$ preserving Γ (see for instance [MM22, §1.1] or [Sma67, Ham13] for a description of such algebraic examples).

Actually, the Theorem A does not rely on any uniformity concerning the contraction (respectively expansion) of E^s (resp. E^u) by $\mathrm{D}f$. More precisely, let f be a diffeomorphism of a three-dimensional closed manifold M having a dense orbit in M (this replaces the hypothesis $NW(f) = M$), preserving a smooth splitting $\mathrm{T}M = E^\alpha \oplus E^c \oplus E^\beta$ with $E^\alpha \oplus E^\beta$ a contact distribution, and assume that for any $x \in M$ we have, for $\varepsilon = \alpha$ and $\varepsilon = \beta$:

$$\lim_{n\to+\infty} \|\mathrm{D}_x f^n|_{E^\varepsilon}\| = 0 \text{ or } \lim_{n\to-\infty} \|\mathrm{D}_x f^n|_{E^\varepsilon}\| = 0 \tag{1.2}$$

with respect to some Riemannian metric on M. Then the conclusions of Theorem A hold on f (see [MM22, Theorem B]). Let us emphasize that the assumption (1.2) is related to, though different from, the notion of *quasi-Anosov diffeomorphism* of Mañé in [Mañ77].

1.3. **Path geometries and Cartan geometries.** The triplet $\mathcal{S} = (E^s, E^c, E^u)$ preserved by a partially hyperbolic diffeomorphism f of contact type as in Theorem A happens to be a *rigid geometric structure*, and the rough idea is that the dynamical properties of the *automorphism* f of $\mathcal{S}$ will imply a geometrical classification of $\mathcal{S}$, giving in return a dynamical classification of f.

On a three-dimensional manifold, a pair $\mathcal{L} = (E^\alpha, E^\beta)$ of transverse line fields whose sum is a contact distribution is called a *path geometry.* These structures are intimately linked with the homogeneous space $\mathbf{X}$ of *full flags of* $\mathbb{R}^3$, endowed with a natural path geometry invariant under the natural action of $\mathrm{PGL}_3(\mathbb{R})$ on $\mathbf{X}$. The flag space $\mathbf{X}$ plays for path geometries the role played by the euclidean space for Riemannian metrics: it is the *flat model.* The notion of *Cartan geometry* (originally due to Élie Cartan, see [Car10, Sha97, ČS09]) allows indeed to give a precise meaning to the following idea: every three-dimensional path geometry $\mathcal{L}$ is a "curved version" of the homogeneous space $\mathbf{X}$, and enjoys a *curvature* whose vanishing is equivalent to $\mathcal{L}$ being *flat*,

[1]Two flows are *orbit equivalent* if there exists a diffeomorphism conjugating their orbits.

[2]The denomination partially hyperbolic actually refers in the litterature to the case where the invariant splitting $E^s \oplus E^c \oplus E^u$ is furthermore *dominated.* This assumption being however unnecessary in Theorem A and elsewhere in this text, we allow ourselves to elude it to simplify the terminology, and refer the interested reader to [CP15].

i.e. locally isomorphic to $\mathbf{X}$. The tools of Cartan geometries play a crucial role in the classification of Theorem A.

In Theorem A, even if the diffeomorphisms are only assumed to preserve the triplet (E^s, E^u, E^c), the classification shows *a posteriori* that they preserve in fact a *global* contact form θ of kernel $E^s \oplus E^u$. In other words, they preserve the triplet $\mathcal{T} = (E^s, E^u, \theta)$, that we call a *strict path structure*. In [GD91], a general program was introduced for studying, and possibly classifying those compact rigid geometric structures having a non-compact automorphism group. In this direction, we obtained with Elisha Falbel and Jose Miguel Veloso the following result concerning strict path structures.

Theorem B ([FMMV21, Theorem 1.1])**.** *Let $(M, \mathcal{T})$ be a three-dimensional closed and connected strict path structure, whose automorphism group is non-compact and has a dense orbit. Then $(M, \mathcal{T})$ is isomorphic to one of the family of examples appearing in Theorem A.*

1.4. **Compactifications of path geometries.** All the diffeomorphisms of Theorem A are *conservative* (*i.e.* preserve a volume form), and moreover preserve a line field E^c transverse to the contact distribution $E^s \oplus E^u$. A first reasonable problem to understand the diversity of path geometries with large automorphism groups is thus to exhibit path geometries enjoying non-conservative automorphisms that are *non-equicontinuous* (*i.e.* generate a non-compact subgroup of the automorphism group) and moreover *essential*: they preserve no line field transverse to the contact distribution. For any (complete) hyperbolic surface Σ, the unitary tangent bundle $\mathrm{T}^1\Sigma$ is endowed with a natural path geometry $\mathcal{L}_\Sigma$ invariant by the geodesic flow, for which I obtained the following.

Theorem C ([MM21, Theorem A])**.** *Let $g_1, \dots, g_d$ be hyperbolic elements of $\mathrm{PSL}_2(\mathbb{R})$ with pairwise distinct fixed points on the boundary $\partial_\infty \mathbf{H}^2$. Then there exists integers $r_i > 0$ such that the hyperbolic surface $\Sigma = \langle g_1^{r_1}, \dots, g_d^{r_d} \rangle \backslash \mathbf{H}^2$ verifies the following.*

1. *The path geometry $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma)$ admits a compactification $(M, \mathcal{L})$.*
2. *Furthermore, the geodesic flow of $\mathrm{T}^1\Sigma$ extends to a non-equicontinuous, non-conservative and essential automorphism flow of $(M, \mathcal{L})$.*

The first statement of this theorem relies on the study of the action of "Schottky" discrete subgroups of $\mathrm{PGL}_3(\mathbb{R})$ on the flag space $\mathbf{X}$, which provides an independent and elementary proof of the existence of open subsets of the flag space with proper and cocompact action of these Schottky subgroups. These domains of discontinuity, also provided by general results about Anosov representations in [GW12, KLP18, BPS19], are here obtained by constructing explicit fundamental domains for the action. This is done by a precise analysis of the dynamics of $\mathrm{PGL}_3(\mathbb{R})$ on $\mathbf{X}$, allowing to obtain the dynamical properties of the compactified geodesic flow in the second statement.

2. Ongoing and future projects

2.1. **Higher-dimensional partially hyperbolic diffeomorphisms of contact type.** Theorem A can be seen as an analog for partially hyperbolic diffeomorphisms of Ghys classification in [Ghy87] of three-dimensional contact-Anosov flows with smooth invariant distributions. In 1992, Ghys theorem was generalized in higher dimensions by Benoist, Foulon and Labourie in [BFL92]: any contact-Anosov flow with smooth stable and unstable distributions is, up to finite coverings and orbit equivalence, the geodesic flow of a closed locally symmetric Riemannian manifold of strictly negative curvature. It is thus natural to look for a discrete-time analog of this classification, that is for a higher-dimensional analog of Theorem A. We are now interested in partially hyperbolic diffeomorphisms f in any (odd) dimension, having a dense orbit, smooth invariant distributions, and for which $E^s \oplus E^u$ is a contact distribution. In a work in progress with Elisha Falbel, we study this problem under the additional assumption that the diffeomorphism preserves a contact form of kernel $E^s \oplus E^u$. The pair (E^s, E^u) is then again a f-invariant rigid geometric structure, called *Lagrangian contact structure*. But going from the case of Anosov flows to the one of partially hyperbolic diffeomorphisms completely changes the situation. A critical difference is for instance that in the former case, the geometric structure is soon found to be locally homogeneous everywhere, whereas it is so only on a dense open subset of the manifold in the latter (as a consequence of Gromov's "open-dense orbit theorem", see [Gro88]). This requires to use the *Cartan geometry* defined by (E^s, E^u), which happens to be significantly more complex than the three-dimensional one – a major difference being that there are multiple local models in higher dimensions.

2.2. **Rigidity of three-dimensional partially hyperbolic diffeomorphisms.** Ghys actually classifies in [Ghy87] all three-dimensional Anosov flows with smooth stable and unstable distributions. A natural project is then to extend Theorem A by classifying all the three-dimensional partially hyperbolic diffeomorphisms having smooth invariant distributions E^s, E^c and E^u – wether $E^s \oplus E^u$ is contact or not. A first result was obtained in this direction in [CPRH20] under the following strong additional restrictions on the partially hyperbolic diffeomorphism f: $\mathrm{D}f$ reads as a constant (diagonal) matrix in some global frame of vector fields generating (E^s, E^c, E^u), and f has a dense orbit. This result was recently precised in [AM21][3] with a new geometrical proof, which is a motivation to look at the general question with geometrical eyes, *i.e.* to consider (E^s, E^c, E^u) as a geometric structure whose behaviour differs between the open subset $O \subset M$ where $E^s \oplus E^u$ is contact and its complement. The case $O = M$ is the one of contact-type partially hyperbolic diffeomorphisms classified in Theorem A. The only known examples for which O is a strict open subset of M are $\mathcal{C}^\infty$-conjugated to the suspension of an Anosov automorphism of the two-torus, in which case the distribution $E^s \oplus E^u$ is integrable and O is thus empty. This suggests that $O \neq \varnothing$ should imply that $O = M$: "if $E^s \oplus E^u$ is contact somewhere, it is contact everywhere" – a conjecture supported by several geometrical evidences. This is a particular case of a more general problem that interests me, that of a degenerating geometric structure defined by invariant distributions – here, the path geometry $\mathcal{L} = (E^s, E^u)$ degenerates on the boundary ∂O.

2.3. **Path geometries and compactifications.** The examples constructed in Theorem C are a motivation to construct new *flat* three-dimensional path geometries, *i.e.* $(\mathrm{PGL}_3(\mathbb{R}), \mathbf{X})$-structures with the language of Ehresman-Thurston (see for instance [Thu97, Chapter 3]). Barbot constructed in [Bar10] a large family of such structures whose holonomies are Anosov representations of a surface group into $\mathrm{PGL}_3(\mathbb{R})$. In [FT15], Falbel and Thebaldi develop a general method of construction of such structures (*via* gluings of tetrahedra in $\mathbf{X}$), which can be seen as an analog for path geometries of Thurston's construction of hyperbolic structures on the complement of a knot in $\mathbf{S}^3$ (see [Thu97] again), and obtain by this way a $(\mathrm{PGL}_3(\mathbb{R}), \mathbf{X})$-structure on an open complete hyperbolic three-manifold. The existence of such a structure on a *closed* hyperbolic three-manifold is an open problem so far, and we would like to understand, in collaboration with Elisha Falbel, if the methods of compactification developed in [MM21] could allow to construct such a structure by "compactifying" the one of [FT15].

2.4. $\mathbb{R}$**-covered Anosov flows.** A natural and important geometrical object associated to an Anosov flow (φ^t) on a three-dimensional closed manifold M, is the *weak stable foliation* $\mathcal{F}^{sc}$, tangent to the plane distribution $E^s \oplus \mathbb{R}\frac{d\varphi^t}{dt}$. Once lifted in the universal cover $\tilde{M}$, it always becomes a foliation by planes whose space of leaves $\mathcal{Q}^s$ is a simply connected one-dimensional manifold which is in general *non-Hausdorff*. Anosov flows for which $\mathcal{Q}^s$ is a line distinguish thus themselves has exceptional ones, and are called $\mathbb{R}$-*covered* Anosov flows. Barbot showed in [Bar01] that *contact-Anosov flows*, *i.e.* Anosov flows for which $E^s \oplus E^u$ is contact, are $\mathbb{R}$-covered. In fact besides those, the only other known examples of $\mathbb{R}$-covered Anosov flows are the trivial ones, which are the suspensions of hyperbolic automorphisms of the two-torus. This is why a long-standing conjecture states that $\mathbb{R}$-covered Anosov flows that are not conjugated to suspensions are orbit-equivalent to contact-Anosov flows. This is the problem that we address in a work in progress with Federico Salmoiraghi and Mario Shannon.

2.5. **Topological characterization of conservative flows in dimension three.** The works of Asimov [Asi76] and Sullivan [Sul76] gave sufficient conditions for a non-singular flow to be isotopic to a *conservative* one. These results leave however open the question of a topological condition for such flows to be *orbit-equivalent* to a conservative one. A natural obstruction in dimension three is the existence of a separating torus transverse to the flow, and results of Brunella [Bru93] and Asaoka [Asa08] show that this is indeed the only obstruction for Anosov flows. It is likely that

[3]In [AM21], the framing only needs to be $\mathcal{C}^1$ instead of $\mathcal{C}^2$ and the topological transitivity assumption is dropped.

it remains the only one for any non-singular three-dimensional flow, which is the subject of an ongoing project in collaboration with Tali Pinsky.

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