GEOMETRICAL COMPACTIFICATIONS OF GEODESIC FLOWS AND PATH STRUCTURES

MARTIN MION-MOUTON

Abstract. In this paper, we construct a geometrical compactification of the geodesic flow of non-compact complete hyperbolic surfaces Σ without cusps having finitely generated fundamental group. We study the dynamical properties of the compactified flow, for which we show the existence of attractive circles at infinity. The geometric structure of $\mathrm{T}^1\Sigma$ for which this compactification is realized is the pair of one-dimensional distributions tangent to the stable and unstable horocyles of $\mathrm{T}^1\Sigma$. This is a Kleinian path structure, that is a quotient of an open subset of the flag space by a discrete subgroup Γ of $\mathrm{PGL}_3(\mathbb{R})$. Our study relies on a detailed description of the dynamics of $\mathrm{PGL}_3(\mathbb{R})$ on the flag space, and on the construction of an explicit fundamental domain for the action of Γ on its maximal open subset of discontinuity in the flag space.

1. Introduction

The geodesic flow (g^t) of a compact hyperbolic surface Σ has a very nice and well-studied dynamical property: it is an *Anosov flow* of its unitary tangent bundle $\mathrm{T}^1\Sigma$. This means that $\mathrm{D}g^t$ preserves a splitting

$$\mathrm{T}(\mathrm{T}^1\Sigma) = E^s \oplus \mathbb{R}\frac{dg^t}{dt} \oplus E^u$$

of the tangent bundle of $\mathrm{T}^1\Sigma$, where $\mathbb{R}\frac{dg^t}{dt}$ is the direction of the flow, and E^s, E^u are two one-dimensional distributions of $\mathrm{T}^1\Sigma$ (respectively called the *stable* and *unstable* distributions of (g^t)) that are respectively *uniformly contracted* and *uniformly expanded* by $\mathrm{D}g^t$. More precisely, for any Riemannian metric on $\mathrm{T}^1\Sigma$, there exists two constants $C > 0$ and $\lambda < 1$ such that for any $x \in \mathrm{T}^1\Sigma$ and $t > 0$:

$$\left\|\mathrm{D}_x g^t|_{E^s}\right\| \leq C\lambda^t \text{ and } \left\|\mathrm{D}_x g^{-t}|_{E^u}\right\| \leq C\lambda^t. \tag{1.1}$$

Geodesic flows of compact hyperbolic surfaces are very specific among Anosov flows, since their stable and unstable distributions are smooth (that is, $\mathcal{C}^\infty$). The sum $E^s \oplus E^u$ moreover happens to be a *contact distribution* – we say in this case that the flow is *contact-Anosov*. We recall that a $\mathcal{C}^1$ plane distribution of a three-dimensional manifold is called contact if it is locally the kernel of a one-form θ which is *contact* ($\theta \wedge d\theta$ nowhere vanishes). A pair $\mathcal{L} = (E^\alpha, E^\beta)$ of smooth one-dimensional distributions whose sum is a contact distribution defines on a three-dimensional manifold a geometric structure called a *path structure*. Path structures, whose study goes back to Élie Cartan in [Car24], are *rigid* geometries whose interplay with smooth dynamics has shown to be very rich (see Paragraph 1.2 for more details about path structures and their rigidity, and Paragraph 1.1.2 for examples explaining the geometrical origin of the terminology). From a geometrical point of view, we may thus look at the geodesic flow of a compact hyperbolic surface Σ as a flow of automorphisms of the path structure $\mathcal{L}_\Sigma = (E^s, E^u)$ on $\mathrm{T}^1\Sigma$. This point of view allowed for instance Ghys to classify in [Ghy87] the three-dimensional contact-Anosov flows having smooth stable and unstable distributions (see Paragraph 1.2 for more details).

One can ask what remains of this beautiful geometrico-dynamical picture for a *non-compact* complete hyperbolic surface Σ. The (g^t)-invariant path structure $\mathcal{L}_\Sigma$ persists in this case, and one of the motivation of this paper is to provide with a geometrico-dynamical compactification of both the structure $\mathcal{L}_\Sigma$ and the flow (g^t) on $\mathrm{T}^1\Sigma$, and to describe the dynamics of the compactified geodesic flow obtained in this way.

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1.1. **Geometrical compactification of the geodesic flow.** The geometric picture that we described is actually independent of the compactness of Σ. Indeed for any complete hyperbolic surface Σ, there exists on $\mathrm{T}^1\Sigma$ a natural path structure $\mathcal{L}_\Sigma = (E^s, E^u)$ invariant by the geodesic flow (g^t) – and equal to the pair of stable and unstable distributions of (g^t) if Σ is compact (see Paragraph 4.1 for a proper definition of $\mathcal{L}_\Sigma$). The hyperbolic metric of Σ induces on $\mathrm{T}^1\Sigma$ a "most natural" Riemannian metric (invariant by the lifts of isometries of Σ) called the *Sasaki metric*, with respect to which (g^t) indeed satisfies the Anosov conditions (1.1) on the distributions E^s and E^u of the path structure $\mathcal{L}_\Sigma$ – wether Σ is compact or not. In this regard, one may thus say that (g^t) is "*Anosov for the Sasaki metric*", and that the path structure $\mathcal{L}_\Sigma$ that we are studying is the pair of stable and unstable distributions of (g^t). An important distinction to be made is however that in the non-compact case, the existence of the inequalities (1.1) will critically depend on the chosen Riemannian metric, since $\mathrm{T}^1\Sigma$ is non-compact (whereas only constants will differ if Σ is compact). The Anosov property is thus in the non-compact case not an intrinsic property of the *flow* (g^t) itself but only of the *pair* $((g^t)$, Sasaki metric). For this reason, we would like to study (g^t) as acting on an open subset of a closed three-manifold. More precisely, we would like to find a *closed* three-manifold M together with a flow (φ^t) on M, containing a (φ^t)-invariant open subset $N \subset M$ such that $(\varphi^t|_N)$ is conjugated to (g^t). In this case, we will say that (M, φ^t) is a (dynamical) *compactification* of $(\mathrm{T}^1\Sigma, g^t)$. In general, it is not clear if a given flow acting on an open manifold can be compactified in that way; but if it does, then there are certainly a lot of possible compactifications, and one would like to choose one that has interesting properties with respect to the flow. To begin with, we would like to preserve – as far as possible – any information that we already have about this flow.

In our case we do have an additional geometrical information, the (g^t)-invariant path structure $\mathcal{L}_\Sigma$ on $\mathrm{T}^1\Sigma$, that we would like to preserve in order to stay as close as possible to the Anosov behaviour. In other words, what we want is a path structure $\mathcal{L} = (E^\alpha, E^\beta)$ on M and an open subset $N \subset M$, such that $(N, \mathcal{L}|_N)$ is isomorphic to $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma)$ – an isomorphism of path structures being simply a diffeomorphism sending E^α (respectively E^β) on E^s (resp. E^u). In this case, we will say that $(M, \mathcal{L})$ is a (path structure) *compactification* of $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma)$. If we moreover ask the dynamics and the geometry to be compatible, then we look for a *closed* three-manifold M endowed with a path structure $\mathcal{L}$ and with a flow (φ^t) of automorphisms of $\mathcal{L}$, such that there exists a (φ^t)-invariant open subset $N \subset M$ and an isomorphism from $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma)$ to $(N, \mathcal{L}|_N)$ conjugating (g^t) and $(\varphi^t|_N)$. In this case, we will say that $(N, \mathcal{L}_N, \varphi^t|_N)$ is *a copy of* $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma, g^t)$ and that $(M, \mathcal{L}, \varphi^t)$ is a *geometrico-dynamical compactification* of $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma, g^t)$.

The following result applies to any hyperbolic surface which is *uniformized by a Schottky subgroup with sufficiently large generators* in the following meaning: for any hyperbolic elements $h_1, \dots, h_d$ of $\mathrm{PSL}_2(\mathbb{R})$ having pairwise distincts fixed points on the boundary of the hyperbolic plane $\mathbf{H}^2$, and for any sufficiently large $r_i > 0$, the statement applies to the quotient of $\mathbf{H}^2$ by the discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ generated by $h_1^{r_1}, \dots, h_d^{r_d}$.

Theorem A. *For any hyperbolic surface Σ uniformized by a Schottky subgroup of $\mathrm{PSL}_2(\mathbb{R})$ with sufficiently large generators, we have the following.*

1. $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma, g^t)$ *admits a geometrico-dynamical compactification $(M, \mathcal{L}, \varphi^t)$, containing four disjoint copies $\{N_i\}_{i=1}^4$ of $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma, g^t)$ and such that $M \setminus N$ is a finite union of tori, with $N = \cup_{i=1}^4 N_i$.*
2. *The set of fixed points of (φ^t) can be decomposed as a disjoint union $\mathcal{C}^- \cup \Delta \cup \mathcal{C}^+$, each of these subsets being a finite union of circles. The subset $\mathcal{W}^+$ (respectively $\mathcal{W}^-$) of points of N whose positive (resp. negative) φ^t-orbit escapes from any compact subset of N is open and dense in N. Furthermore for any $x \in \mathcal{W}^+$ (resp. $x \in \mathcal{W}^-$), $\varphi^t(x)$ converges to a point of $\mathcal{C}^+$ (resp. $\varphi^{-t}(x)$ converges to a point of $\mathcal{C}^-$) when $t \to +\infty$. More precisely, compact subsets of $\mathcal{W}^\pm$ are attracted to $\mathcal{C}^\pm$ under $\varphi^{\pm t}$.*
3. *The support of any (φ^t)-invariant Borel probability measure on M is contained in $M \setminus (\mathcal{W}^+ \cap \mathcal{W}^-)$, and has in particular empty interior.*

Remark 1.1. All the hyperbolic surfaces considered in Theorem A are non-compact complete hyperbolic surfaces of infinite volume without cusps (Σ only has funnels), and with finitely generated fundamental group. Moreover, for any connected non-compact topological surface S with finitely generated fundamental group, the set of hyperbolic metrics g on S for which Theorem A applies to $\Sigma = (S, g)$ is open and non-empty.

Theorem A will be proved in section 4. More precisely, refined versions of the three claims are respectively proved in Propositions 4.7, 4.9, 4.10 and Corollary 4.11.

1.1.1. *About unicity of compactifications.* It may seem surprising that the compactification $(M, \mathcal{L}, \varphi^t)$ of the geodesic flow given by Theorem A contains *four* copies of $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma, g^t)$, and one may ask if there exists a "smaller" compactification. In particular, it is natural to ask:

Question a. Does there exist a geometrico-dynamical compactification $(M, \mathcal{L}, \varphi^t)$ containing a dense copy of $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma, g^t)$? Or a path structure compactification $(M, \mathcal{L})$ containing a dense copy of $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma)$?

We will discuss in Paragraph 1.3 below a partial answer to this question when restricted to *Kleinian* compactifications. Another surprising property of the compactification given by Theorem A is the existence of circles of fixed points for the compactified geodesic flow (φ^t). This raises the following second question, intimately linked to the previous one.

Question b. Does there exist a geometrico-dynamical compactification $(M, \mathcal{L}, \varphi^t)$ of $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma, g^t)$ where (φ^t) has no fixed point ? Or, at least, where all its fixed points are isolated ?

1.1.2. *Other geometrical compactifications of* $\mathrm{T}^1\Sigma$*.* The unitary tangent bundle of a hyperbolic surface Σ actually bears different geometric structures, and it is interesting to compare the compactification obtained in Theorem A for the path structure $\mathcal{L}_\Sigma$ with those obtained for other structures. First of all, a path structure is associated to any Riemannian surface S in the following way. The set of geodesics of S defines on T^1S a one-dimensional distribution E^β tangent to the lifts of the geodesics in T^1S. Denoting by E^α the tangent direction the fibers of the canonical projection $\pi\colon \mathrm{T}^1S \to S$, $E^\alpha \oplus E^\beta$ is then a contact distribution. In other words the unitary tangent bundle T^1S of any Riemannian surface is naturally endowed with a path structure $\mathcal{L}_S^{proj} = (E^\alpha, E^\beta)$, wether S is hyperbolic or not. These classical examples explain the geometrical origin of the terminology *path structure*.

In the specific case of a complete hyperbolic surface Σ, we thus have *two different* path structures $\mathcal{L}_\Sigma$ and $\mathcal{L}_\Sigma^{proj}$ on $\mathrm{T}^1\Sigma$. The main difference between those two structures is that $\mathcal{L}_\Sigma^{proj}$ *is not invariant by the geodesic flow* (g^t). Indeed, E^β is (g^t)-invariant by definition (as it is tangent to the orbits of (g^t)), but a fiber of π is not sent by g^t to another fiber and E^α is thus not (g^t)-invariant. We will explain in Paragraph 4.5 how the work of Choi-Goldman in [CG17] gives a compactification of $\mathcal{L}_\Sigma^{proj}$, and we will describe the conformal compactification given by [Fra05] of a g^t-invariant Lorentzian metric on $\mathrm{T}^1\Sigma$.

1.2. **Path structures with non-compact automorphism groups and partially hyperbolic diffeomorphisms.** The initial motivation of Élie Cartan for the study of path structures in [Car24] was to find a geometrical object that parametrizes the space of local solutions of second-order scalar ordinary differential equations (see for instance [IL, §8.6] for an explanation of this link), and to describe the local invariants of such an ODE through a notion of *curvature* of path structures. Path structures are nowadays studied in a geometric setting called *parabolic Cartan geometries* and are sometimes called *Lagrangian-contact structures* in this context (see for instance [ČS09, §4.2.3], [Tak94]). The author actually used the denomination Lagrangian-contact structure in [MM21] before deciding to stick to the name *path structure* which seems more geometrically meaningful and closer to the initial motivation of Cartan. We apologize in advance for any confusion that this change of name could lead to.

Apart from the intrinsic interest of compactification of geodesic flows, a second important motivation of this paper was to deduce from Theorem A new and rich examples of path structures having non-compact automorphism groups for the compact-open topology. The interest of such examples appears in contrast with former rigidity results for path structures that we now describe.

1.2.1. *Hierarchy of path structures.* Ghys used the path structures to prove in [Ghy87] that the geodesic flows of compact hyperbolic surfaces are the only three-dimensional contact-Anosov flows having smooth stable and unstable distributions, up to finite coverings and smooth orbit-equivalence (path structures appear in [Ghy87] through the point of view of second order ODE mentionned previously). In fact such a contact-Anosov flow (φ^t) preserves more than the path structure (E^s, E^u) defined by its stable and unstable distributions. The contact form θ defined by $\theta(\frac{d\varphi^t}{dt}) \equiv 1$ and $\theta|_{E^s \oplus E^u} \equiv 0$ is indeed (φ^t)-invariant, and (φ^t) preserves thus the triplet (E^s, E^u, θ). This is a special instance of a *strict path structure* $\mathcal{T} = (E^\alpha, E^\beta, \theta)$, with (E^α, E^β) a path structure and θ a contact form of kernel $E^\alpha \oplus E^\beta$. From a geometrical point of view, Ghys result corresponds thus to classify the compact strict path structures whose Reeb flow is Anosov. In [FMMV21] we generalize this result with Elisha Falbel and Jose Miguel Veloso by considering the three-dimensional compact strict path structures $(M, \mathcal{T})$ having a non-compact automorphism group and a dense Aut^{loc}-orbit. We prove in this setting that up to finite coverings, $(M, \mathcal{T})$ is either the structure preserved by the geodesic flow of a compact hyperbolic surface, or a left-invariant structure on a compact quotient of the Heisenberg group Heis(3).

One can now forget about the contact form θ to keep only a third direction E^c transverse to the contact distribution $E^\alpha \oplus E^\beta$ and consider the triplet $\mathcal{S} = (E^\alpha, E^\beta, E^c)$ that we will call an *enhanced path structure* (this can be considered as the “conformal version” of a strict path structure, the Reeb vector field of the contact form being weakened to a line field). We can then ask the same question: what are the three-dimensional compact enhanced path structures $(M, \mathcal{S})$ having a non-compact automorphism group ? In [MM21] we obtain a first classification result in this direction, assuming that an automorphism f of $\mathcal{S}$ without wandering points uniformly contracts or expands both E^α and E^β. In this case we prove that $(M, \mathcal{S})$ still belongs to one of the two families of algebraic examples previously mentionned (geodesic flows or compact quotients of Heis(3)) which yields a rigidity result about partially hyperbolic diffeomorphisms, see [MM21, Theorem A]. In particular, there exists *a posteriori* a contact form θ such that any automorphism of the enhanced path structure $\mathcal{S}$ is in fact an automorphism of the *strict* path structure $(E^\alpha, E^\beta, \theta)$.

The next step would be to forget the transverse direction E^c and to investigate the three-dimensional compact path structures $\mathcal{L} = (E^\alpha, E^\beta)$ having a non-compact automorphism group. Until now, we saw two kinds of examples of automorphisms of path structures generating a non-compact subgroup, which we will call *non-equicontinuous automorphisms*: the first are time-one maps of geodesic flows of compact hyperbolic surfaces, and the second are automorphisms of compact quotients of Heis(3). In particular, all of these examples are partially hyperbolic diffeomorphisms (see [MM21, Theorem A]) and are *conservative* (they preserve a volume form). Other examples are easily constructed in the following way. Take φ an automorphism of Heis(3) which is diagonal and expands the three directions in an affine chart of Heis(3). Then in the same way as classical Hopf tori, the quotient $\langle\varphi\rangle\backslash\mathrm{Heis}(3)$ is compact and bears a path structure $\mathcal{L}$ having non-conservative automorphisms (see [Ale21, p.24] for more details, and for links with completeness results about flat strict path structures). These examples have a simple geometrical and topological description, all of them are homeomorphic to $\mathbf{S}^1 \times \mathbf{S}^2$. Moreover, all their automorphisms preserve not only the path structure $\mathcal{L}$ but also a direction E^c transverse to $\mathcal{L}$.

1.2.2. *New essential path structures.* These last examples suggest that, in order to find “more complicated” examples of path structures $\mathcal{L}$, one should look for automorphisms of $\mathcal{L}$ that do not preserve any smooth one-dimensional distribution transverse to the contact distribution of $\mathcal{L}$ – or in other words, that do not preserve any enhanced path structure compatible with $\mathcal{L}$. We will say that an automorphism flow is *strongly essential* if it does not preserve any continuous distribution transverse to $\mathcal{L}$. This notion is reminiscent of the *essential Lorentzian conformal structures*, whose conformal automorphism group is the isometry group of no metric in the conformal class. These structures are studied in [Fra05], where essential Lorentzian conformal structures distinct from the Einstein universe are constructed. One of the motivation of this paper is to provide with the following large family of new examples of compact path structures with essential automorphisms.

Theorem B. *Let $(M,\mathcal{L},\varphi^t)$ be the compactification of $(\Sigma,\mathcal{L}_\Sigma,g^t)$ described in Theorem A. Then for any $t\neq 0$, φ^t is non-equicontinuous, non-conservative and not partially hyperbolic. Furthermore, (φ^t) is a strongly essential automorphism flow.*

These claims are proved in Corollary 4.11, and Propositions 4.12 and 4.13.

1.3. **Compactifications of Kleinian path structures.** So far, we considered a non-compact complete hyperbolic surface Σ together with the path stucture $\mathcal{L}_\Sigma$ of $\mathrm{T}^1\Sigma$ invariant by its geodesic flow (g^t), and we described the dynamical properties of the compactified geodesic flow with respect to $\mathcal{L}_\Sigma$. We now explain the geometric origin of this compactification. Recall from Theorem A that we consider a surface Σ which is the quotient of $\mathbf{H}^2$ by a discrete free subgroup $\overline{\Gamma}_0$ of $\mathrm{PSL}_2(\mathbb{R})$ generated by d hyperbolic elements $\overline{h}_i$ of $\mathrm{PSL}_2(\mathbb{R})$ having pairwise distinct fixed points on $\partial\mathbf{H}^2$. We choose for each i a lift $h_i\in\mathrm{SL}_2(\mathbb{R})$ of $\overline{h}_i$ with positive eigenvalues, and we consider the subgroup

$$\Gamma_0=\langle h_1,\dots,h_d\rangle$$

of $\mathrm{SL}_2(\mathbb{R})$ generated by the h_i. It is known that $\mathrm{T}^1\Sigma$ is identified with $\overline{\Gamma}_0\backslash\mathrm{PSL}_2(\mathbb{R})$, and we thus have a two-sheeted covering from $\Gamma_0\backslash\mathrm{SL}_2(\mathbb{R})$ to $\mathrm{T}^1\Sigma$. As we will see in Lemma 4.1, the pullback of the path structure $\mathcal{L}_\Sigma$ of $\mathrm{T}^1\Sigma$ by this covering comes from a left-invariant path structure $\mathcal{L}_{\mathrm{SL}_2(\mathbb{R})}$ on $\mathrm{SL}_2(\mathbb{R})$. It turns out that $(\mathrm{SL}_2(\mathbb{R}),\mathcal{L}_{\mathrm{SL}_2(\mathbb{R})})$ can be embedded in a global *homogeneous model space*, central in the study of path structures.

This is the *flag space* of dimension three that we denote by $\mathbf{X}$, defined as

$$\mathbf{X}=\left\{(p,D)\in\mathbb{R}\mathbf{P}^2\times\mathbb{R}\mathbf{P}^2_*\;\middle|\;p\in D\right\}$$

where $\mathbb{R}\mathbf{P}^2$ (respectively $\mathbb{R}\mathbf{P}^2_*$) denotes the projective plane (resp. the space of projective lines of $\mathbb{R}\mathbf{P}^2$). $\mathbf{X}$ admits two natural projections π_α and π_β, respectively on $\mathbb{R}\mathbf{P}^2$ and $\mathbb{R}\mathbf{P}^2_*$, which are the restrictions to $\mathbf{X}$ of the first and second coordinate projections, and whose fibers define two transverse foliations of $\mathbf{X}$ by circles that we respectively call α and β*-circles.* Denoting respectively by $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ the distributions tangent to these foliations, $\mathcal{E}_\alpha\oplus\mathcal{E}_\beta$ is a contact distribution and $\mathcal{L}_{\mathbf{X}}=(\mathcal{E}_\alpha,\mathcal{E}_\beta)$ is thus a path structure on $\mathbf{X}$. Note that there is a natural identification of $\mathbf{X}$ with $\mathbf{P}(\mathrm{T}\mathbb{R}\mathbf{P}^2)$, for which $\mathcal{L}_{\mathbf{X}}$ is simply the structure $\mathcal{L}^{proj}_{\mathbb{R}\mathbf{P}^2}$ that we defined in Paragraph 4.5.1, considering the projective lines as the geodesics of $\mathbb{R}\mathbf{P}^2$.

The natural action of $\mathrm{PGL}_3(\mathbb{R})$ on $\mathbf{X}$ does not only preserve $\mathcal{L}_{\mathbf{X}}$, but is equal to its whole automorphism group. Note that the action of $\mathrm{PGL}_3(\mathbb{R})$ is transitive and identifies thus $\mathbf{X}$ with the homogeneous space $\mathrm{PGL}_3(\mathbb{R})/\mathbf{P}_{min}$, with $\mathbf{P}_{min}=\mathrm{Stab}([e_1],[e_1,e_2])$ the subgroup of upper-triangular matrices – which is a (minimal) parabolic subgroup of $\mathrm{PGL}_3(\mathbb{R})$. When embedded in $\mathrm{PGL}_3(\mathbb{R})$ through

$$j\colon A\in\mathrm{SL}_2(\mathbb{R})\mapsto\begin{bmatrix}A&0\\0&1\end{bmatrix}\in\mathrm{PGL}_3(\mathbb{R}),$$

$\mathrm{SL}_2(\mathbb{R})$ has an unique open orbit on $\mathbf{X}$ that we denote by Y, for which $j(\Gamma_0)\backslash Y$ is isomorphic to $\Gamma_0\backslash\mathrm{SL}_2(\mathbb{R})$ (see Lemma 4.1). Here the second quotient is endowed with the path structure induced by $\mathcal{L}_{\mathrm{SL}_2(\mathbb{R})}$, and $j(\Gamma_0)\backslash Y$ with the one induced by $\mathcal{L}_{\mathbf{X}}$ – this is a special instance of *Kleinian* path structures, that are quotients of open subsets of $\mathbf{X}$ by discrete subgroups of $\mathrm{PGL}_3(\mathbb{R})$.

Theorem C. *Up to replacing each generator h_i by a large enough finite iterate $h_i^{r_i}$, the Kleinian structure $j(\Gamma_0)\backslash Y$ admits a Kleinian compactification $\Gamma\backslash\Omega$ where it embedds as an open and dense subset. Moreover, $\Gamma\backslash\Omega$ is homeomorphic to a closed three-manifold obtained from the flag space* $\mathbf{X}$ *after performing d times a topological surgery described by the folllowing two operations.*

- *(A) Remove the interior of two disjoint embedded genus two handlebodies H^- and H^+.*
- *(B) Glue the two boundary components ∂H^- and ∂H^+ of the resulting three-manifold with boundary, by some diffeomorphism between ∂H^- and ∂H^+.*

We emphasize that the gluing diffeomorphisms that appear are extremely specific (they arise from elements of $\mathrm{PGL}_3(\mathbb{R})$), and that we actually expect the topology of these surgeries to be highly constrained (it is likely that this topology does only depend on the number of generators of Γ_0). After this result and regarding the above Question a in Paragraph 1.1.1, it seems even more surprising that the compactification of $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma)$ in Theorem A contains four copies of $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma)$, while its two-sheeted covering $\Gamma_0\backslash\mathrm{SL}_2(\mathbb{R})$ admits a compactification with a dense copy. One can actually prove that the answer to Question a is negative if the path structure compactification $(M, \mathcal{L})$ is assumed to be Kleinian. However there is *a priori* no reason for a path structure compactification of $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma)$ to be Kleinian, and obtaining a complete answer to Question a is thus much more difficult than handling the specific case of Kleinian compactifications.

Theorem C is proved in Proposition 4.4, and will be a direct consequence of a more general result that we now present.

1.3.1. *Fundamental domains for Schottky subgroups.* We will call *loxodromic* any diagonalizable element of $\mathrm{PGL}_3(\mathbb{R})$ having three eigenvalues of distinct absolute values. A loxodromic element $g \in \mathrm{PGL}_3(\mathbb{R})$ acts particularly nicely on the flag space: there exists a repulsive bouquet of two circles $B^-_{\alpha\beta}(g) \subset \mathbf{X}$ and an attractive bouquet of two circles $B^+_{\alpha\beta}(g) \subset \mathbf{X}$ with respect to which the dynamics of (g^n) are of "north-south type", meaning that any compact subset of $\mathbf{X} \setminus B^-_{\alpha\beta}(g)$ converges to $B^+_{\alpha\beta}(g)$ under the action of (g^n) (see Example 2.22 for more details). For any g, $B^\pm_{\alpha\beta}(g)$ is the bouquet of α and β-circles of a point $x^\pm \in \mathbf{X}$, and these g-invariant bouquets of circles play for g the role of the attractive and repulsive fixed points in $\partial\mathbf{H}^2$ of an hyperbolic element of $\mathrm{PSL}_2(\mathbb{R})$. A natural analog to the classical definition of Schottky subgroups of $\mathrm{PSL}_2(\mathbb{R})$ is then the following.

Definition 1.2. The group Γ generated by loxodromic elements $g_1, \dots, g_d \in \mathrm{PGL}_3(\mathbb{R})$ is a *Schottky subgroup* if there exists a *set of separating handlebodies* $\{H_i^-, H_i^+\}_{i=1}^d$ for the g_i, where the $H_i^\pm$ are pairwise disjoint compact neighbourhoods of the $B^\pm_{\alpha\beta}(g_i)$ in $\mathbf{X}$ that are genus two handlebodies, such that $H_i^+ = \mathbf{X} \setminus \mathrm{Int}(g_i(H_i^-))$ for any i.

In particular Schottky subgroups of $\mathrm{PGL}_3(\mathbb{R})$ are free groups, and as in $\mathrm{PSL}_2(\mathbb{R})$ we will see in Proposition 3.4 that for any loxodromic elements $g_1, \dots, g_d \in \mathrm{PGL}_3(\mathbb{R})$ *in general position*, that is whose bouquet of circles $B^\pm_{\alpha\beta}(g_i)$ are pairwise disjoint, $\Gamma = \langle g_1, \dots, g_d\rangle$ is a Schottky subgroup up to replacing each g_i by $g_i^{r_i}$ for $r_i > 0$ large enough. Any sequence of $\mathrm{PGL}_3(\mathbb{R})$ eventually escaping from any compact has a subsequence going *simply* to infinity in a sense defined in Definition 2.3. Among those, the sequences (γ_n) of *balanced type* (see Definition 2.4) have on $\mathbf{X}$ the same kind of north-south dynamics than the iterates of a loxodromic element, with respect to a repulsive and an attractive bouquet of two circles $B^\pm_{\alpha\beta}(\gamma_n)$ (see Lemma 2.21 for more details).

Theorem D. *Let Γ be a Schottky subgroup of $\mathrm{PGL}_3(\mathbb{R})$ with d generators.*

1. *Any sequence $\gamma_n \in \Gamma$ going simply to infinity is of balanced type, and Γ acts freely, properly and cocompactly on the open subset:*
$$\Omega(\Gamma) = \mathbf{X} \setminus \bigcup_{\substack{\gamma_n \in \Gamma \\ \gamma_n \underset{simply}{\longrightarrow} \infty}} B^+_{\alpha\beta}(\gamma_n).$$
2. *With $\{H_i^-, H_i^+\}_{i=1,\dots,d}$ a set of separating handlebodies for the g_i, $\mathbf{X} \setminus \bigcup_i (H_i^- \cup H_i^+)$ is a fundamental domain for the action of Γ on $\Omega(\Gamma)$.*
3. *The topology of $\Gamma\backslash\Omega(\Gamma)$ is obtained from the flag space $\mathbf{X}$ after performing d times the surgery described in Theorem C.*
4. *In the case of $d = 1$ loxodromic element with positive eigenvalues, $\Omega(\Gamma)$ is the complement in $\mathbf{X}$ of two disjoint bouquet of circles, and $\Gamma\backslash\Omega(\Gamma)$ is homeomorphic to the product of the circle with the closed connected and orientable surface of genus two.*

The existence of the open subset $\Omega(\Gamma) \subset \mathbf{X}$ with proper and cocompact action of Γ is a consequence of general theories independently developped by [GW12] and [KLP17] for Anosov representations and CEA subgroups, as we will explain in the next paragraph. The interpretation that we give of the boundary $\partial\Omega(\Gamma)$ as the union of the attractive bouquets of two circles of the sequences of Γ going simply to infinity does not appear in this form in these works, though being closely related to the descriptions given therein (see respectively [GW12, §10.2.6] and [KLP17,

§6]). We will give in Paragraph 3 of this paper an independent proof of Theorem D. More precisely, we prove the existence of $\Omega(\Gamma)$ and describe $\partial\Omega(\Gamma)$ for Schottky subgroups of $\mathrm{PGL}_3(\mathbb{R})$ in Propositions 3.6 and 3.7. Our proof relies on a simple ping-pong argument using an explicit description of a fundamental domain for the action of Γ on $\Omega(\Gamma)$ (see Corollary 3.10), which allows us to describe the topology of $\Gamma\backslash\Omega(\Gamma)$ by a surgery (see also Proposition 3.2 for the case of one generator).

1.3.2. *Relations with Anosov representations.* It follows from [BPS19, Theorem 5.9] and independently from the serie of papers [KLP14, KLP16, KLP18] (see also related criteria in [GGKW17]) that if $\Gamma = \langle g_1, \dots, g_d \rangle$ is a Schottky subgroup of $\mathrm{PGL}_3(\mathbb{R})$ in the sense of Definition 1.2, then the induced representation of the free group with d generators into $\mathrm{PGL}_3(\mathbb{R})$ is *Anosov*. The notion of Anosov representation, originally introduced by Labourie in [Lab06], has been intensively studied in the past years. In particular, for Anosov representations of finitely generated word-hyperbolic groups in a semi-simple Lie group G, [GW12] proves the existence of a Γ-invariant open subset Ω of an homogeneous space G/P where the action of Γ is properly discontinuous and cocompact. Independently, [KLP17] proves analog results for *CEA* subgroups of G – a notion closely linked to the one of Anosov representations. Schottky subgroups of $\mathrm{PGL}_3(\mathbb{R})$ fall into both settings (see [KLP17, Remark 1.6]) and the existence of the open subset of discontinuity in $\mathbf{X} = \mathrm{PGL}_3(\mathbb{R})/\mathbf{P}_{min}$ with cocompact action of Γ appearing in Theorem D is a particular case of the results [GW12, Theorem 1.11] and [KLP17, Theorem 1.8]. We also point out [ST18], where the authors extend some of these results to the setting of *purely hyperbolic generalized Schottky subgroups* of $\mathrm{PSL}_{2n+1}(\mathbb{R})$ acting on spaces of oriented flags.

In [Bar01, Bar10], Barbot studies the case of Anosov representations of fundamental groups of closed higher genus surfaces into $\mathrm{PGL}_3(\mathbb{R})$ – actually, the work [Bar01] precedes the definition of Anosov representations. In the same way, the paper [Fra04] studies the action of conformal Schottky subgroups on the Einstein universe before the general investigation of Anosov representations. Both of these works were important inspirations for the point of view adopted in Paragraph 3 of this paper on Schottky subgroups of $\mathrm{PGL}_3(\mathbb{R})$.

1.4. **Dynamics of $\mathrm{PGL}_3(\mathbb{R})$ on the flag space.** The central tool of this paper, for the dynamical results proved in section 4 and for the construction of fundamental domains for Schottky subgroups in section 3, is a detailed description of the dynamics of $\mathrm{PGL}_3(\mathbb{R})$ on the flag space $\mathbf{X}$. This is the content of section 2, which is of independent interest. For any sequence (g_n) going simply to infinity in $\mathrm{PGL}_3(\mathbb{R})$, we describe two "dual" filtrations by natural geometric objects of $\mathbf{X}$, that are pairwise repulsive and attractive objects for the action of (g_n). This description is achieved in Paragraph 2.5 by a very hands-on approach based on the description of the dynamics of $\mathrm{PGL}_3(\mathbb{R})$ on $\mathbb{R}\mathbf{P}^2$ in Paragraph 2.3.

These results are related to those obtained in [KLP17, §6] in the general setting of *regular* discrete subgroups of semi-simple Lie groups – the specificity of the case that we consider in this paper allowing us to obtain a refined geometrical description.

We also point out [FT15] where the authors construct a $(\mathrm{PGL}_3(\mathbb{R}), \mathbf{X})$-structure (called a *flag structure* in their paper) on an open hyperbolic three-manifold. The existence of such a structure on a *closed* hyperbolic three-manifold is an open question, and a natural way to address it would be to compactify the flag structure of [FT15] by using the dynamics of $\mathrm{PGL}_3(\mathbb{R})$ on $\mathbf{X}$.

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Notations and conventions. We denote by $\mathrm{Diag}(\alpha_1, \dots, \alpha_n)$ the diagonal $(n\times n)$-matrix whose entries are the $\alpha_i \in \mathbb{R}$ (or its projection in $\mathrm{PGL}_n(\mathbb{R})$, the context avoiding any confusion), and by $[g]$ the class in $\mathrm{PGL}_n(\mathbb{R})$ of an element $g \in \mathrm{GL}_n(\mathbb{R})$. We denote $[x_1 : \dots : x_n] = \mathbb{R}(x_1, \dots, x_n) \in \mathbb{R}\mathbf{P}^{n-1}$ for any $(x_1, \dots, x_n) \neq (0, \dots, 0)$ and $[P]$ denotes the projection in $\mathbb{R}\mathbf{P}^{n-1}$ of $\mathrm{Vect}(P)$ for any any $P \subset \mathbb{R}^n$. The standard basis of $\mathbb{R}^n$ will always be denoted by $(e_1, \dots, e_n)$.

Any differential geometric object will be assumed to be smooth (that is, $\mathcal{C}^\infty$) if not otherwise specified, and all manifolds will be assumed to be boundaryless.

2. Dynamics in the flag space

The *flag space* is defined by

$$\mathbf{X} = \{(p, D) \mid p \in D\} \subset \mathbb{R}\mathbf{P}^2 \times \mathbb{R}\mathbf{P}^2_*, \tag{2.1}$$

where $\mathbb{R}\mathbf{P}^2$ denotes the projective space and $\mathbb{R}\mathbf{P}^2_*$ the space of projective lines of $\mathbb{R}\mathbf{P}^2$. The natural action of $\mathrm{PGL}_3(\mathbb{R})$ on $\mathbf{X}$ defined by $g\cdot(p, D) = (g(p), g(D))$ is transitive and induces thus an identification of $\mathbf{X}$ with the homogeneous space $\mathrm{PGL}_3(\mathbb{R})/\mathbf{P}_{min}$, with $\mathbf{P}_{min}$ the subgroup of upper-triangular matrices of $\mathrm{PGL}_3(\mathbb{R})$ – which is the stabilizer of $([e_1], [e_1, e_2]) \in \mathbf{X}$. Our goal in this first section is to describe the dynamics of sequences of elements of $\mathrm{PGL}_3(\mathbb{R})$ on $\mathbf{X}$, which will be achieved in Paragraph 2.5.

2.1. Dynamic sets.

Our main technical tool to precisely describe the dynamics of a sequence of diffeomorphisms (g_n) of a compact manifold M, will be the *dynamic sets* of the points $x \in M$, defined as:

$$\mathcal{D}_{(g_n)}(x) = \left\{\text{accumulation points of } (g_n(x_n)) \;\middle|\; (x_n) \in M^{\mathbb{N}}, \lim x_n = x\right\}. \tag{2.2}$$

It is not difficult to check that these are closed and thus compact subsets of M, which are always non-empty since M is compact. The first important utility of dynamic sets is to determine the properness of group actions on open sets of M. For Γ a topological group acting on M and $x \in M$, we denote by $\mathcal{D}_\Gamma(x)$ the union of the $\mathcal{D}_{(g_n)}(x)$, (g_n) being any sequence of Γ *going to infinity* (that is escaping from any compact subset of Γ). We will say that two points x and y of M are *dynamically related* for the action of Γ if $y \in \mathcal{D}_\Gamma(x)$ or $x \in \mathcal{D}_\Gamma(y)$. The following Lemma is then a straightforward translation of the classical definition of proper actions using dynamic sets.

Lemma 2.1. *Γ acts properly on an open set Ω of M, if and only if no pair of points of Ω is dynamically related for the action of Γ.*

See [Fra04, KLP17], where the notions of dynamic sets and dynamical relations are used to prove the properness of different group actions.

Dynamic sets are also an efficient way to determine the limit of a sequence of compact subsets of M. Endowing M with a Riemannian metric and its associated distance d, we recall that the set $\mathcal{K}(M)$ of compact subsets of M is endowed with the classical *Hausdorff distance*

$$d_H(K, L) = \inf \{r > 0 \mid K \subset L_r \text{ and } L \subset K_r\}$$

for any K and L in $\mathcal{K}(M)$, where $K_r = \{x \in M \mid d(x, K) \leq r\}$. The topology induced by this distance on $\mathcal{K}(M)$ is actually independent of the Riemannian metric originally chosen on M, since two such metrics induce bi-Lipschitz equivalent distances by compacity of M. We will always implicitly endow $\mathcal{K}(M)$ with the topology induced in this way by the Hausdorff distance, that we call the *Hausdorff topology*, and for which $\mathcal{K}(M)$ is compact. We denote by $\mathrm{Int}\, P$ the interior of a subset P, and by $\mathrm{Cl}\, P$ its closure.

Lemma 2.2. *Let (g_n) be a sequence of diffeomorphisms of M.*

1. *Let K be a compact subset of M such that $g_n(K)$ converges to a compact K_∞ for the Hausdorff topology. Then $\mathrm{Cl}(\bigcup_{x\in \mathrm{Int}(K)} \mathcal{D}_{(g_n)}(x)) \subset K_\infty \subset \bigcup_{x\in K} \mathcal{D}_{(g_n)}(x)$.*

2. Let K be a compact subset of M of non-empty interior such that

$$\bigcup_{x\in K}\mathcal{D}_{(g_n)}(x)\subset \mathrm{Cl}\left(\bigcup_{x\in\mathrm{Int}(K)}\mathcal{D}_{(g_n)}(x))\right).$$

Then $g_n(K)$ converges to $\bigcup_{x\in K}\mathcal{D}_{(g_n)}(x)$ for the Hausdorff topology.

Proof. 1. Let us denote $K_\infty=\lim g_n(K)$. Concerning the first inclusion, let $y\in\mathcal{D}_{(g_n)}(x)$ with $x\in\mathrm{Int}(K)$, say $y=\lim g_n(x_n)$ with x_n converging to x. Then for n large enough $g_n(x_n)\in g_n(K)$, and thus $d(g_n(x_n),K_\infty)\leq d_H(g_n(K),K_\infty)$. Hence $d(y,K_\infty)=\lim d(g_n(x_n),K_\infty)=0$ since $\lim d_H(g_n(K),K_\infty)=0$ by hypothesis, implying $y\in K_\infty$ as the latter is closed. The inclusion follows since K_∞ is closed. For the second inclusion, let $y\in K_\infty$. Then for any $r>0$, $y\in(g_n(K))_r$ for n large enough. There exists thus n as large as we want and $x_n\in K$, such that $d(y,g_n(x_n))\leq r$. Passing to a subsequence, there exists finally a sequence $x_n\in K$ such that $g_n(x_n)$ converges to y. Possibly taking a further subsequence, we can assume that (x_n) converges to some point $x\in K$, and then $y\in\mathcal{D}_{(g_n)}(x)$, finishing the proof.
2. According to the first claim of the Lemma, if this inclusion holds, then $\cup_{x\in K}\mathcal{D}_{(g_n)}(x)$ is the unique accumulation point of $(g_n(K))$. Since $\mathcal{K}(M)$ is a compact metrizable space, this forces $(g_n(K))$ to converge to K_∞. ☐

Note that, K being compact, a diagonal argument shows that $\bigcup_{x\in K}\mathcal{D}_{(g_n)}(x)$ is closed, so that the inclusion $\mathrm{Cl}(\cup_{x\in\mathrm{Int}(K)}\mathcal{D}_{(g_n)}(x))\subset\cup_{x\in K}\mathcal{D}_{(g_n)}(x)$ always holds. Therefore, the inclusion which is assumed in the second part of the Lemma is actually equivalent to the equality between these two sets.

2.2. **Dynamical types in** $\mathrm{PGL}_3(\mathbb{R})$.

We now come back to the setting that we will be interested with, and describe the three possible asymptotical types of a sequence of $\mathrm{PGL}_3(\mathbb{R})$ going to infinity.

2.2.1. *Cartan decomposition and projection.* Compact perturbations of a sequence (g_n) do not change the nature of its dynamic, but only shift its dynamic sets. More precisely, if two sequences (g_n) and (a_n) of $\mathrm{PGL}_3(\mathbb{R})$ satisfy $g_n=k_na_nl_n$, where (k_n) and (l_n) respectively converge to k_∞ and l_∞ in $\mathrm{PGL}_3(\mathbb{R})$, then

$$\mathcal{D}_{(g_n)}(x)=k_\infty\mathcal{D}_{(a_n)}(l_\infty(x)). \tag{2.3}$$

This relation is a good motivation to reduce the description of the dynamics in $\mathrm{PGL}_3(\mathbb{R})$ to the study of a particularly simple types of elements: diagonal matrices. To this end, $\mathrm{PGL}_3(\mathbb{R})$ enjoys the useful *Cartan decomposition* $\mathrm{PGL}_3(\mathbb{R})=KAK$, with $K\coloneqq\mathrm{PO}(3)$ the orthogonal group and

$$A\coloneqq\left\{\mathrm{Diag}(\alpha,\beta,\gamma)=\begin{bmatrix}\alpha&0&0\\0&\beta&0\\0&0&\gamma\end{bmatrix}\;\middle|\;\alpha,\beta,\gamma>0\right\}\subset\mathrm{PGL}_3(\mathbb{R}) \tag{2.4}$$

the subgroup of diagonal elements of $\mathrm{PGL}_3(\mathbb{R})$ *having positive entries* (note that the KAK decomposition is in this setting a simple consequence of the polar decomposition). We emphasize that in a decomposition $g=kal$ with $(k,l)\in K^2$ and $a\in A$, the pair (k,l) is non-unique but a is unique up to permutation of its diagonal entries which are the singular values of g (that is the squared roots of the eigenvalues of ${}^tgg$). Any $a\in A$ is thus conjugated to an unique *standard element* $\mathrm{Diag}(\alpha,\beta,\gamma)\in A$ such that $\alpha\geq\beta\geq\gamma$. The set of standard elements of A will be denoted by A^+ (this is only a semi-subgroup of A). Any $g\in\mathrm{PGL}_3(\mathbb{R})$ enjoys thus a *standard decomposition*

$$g=kal,\ \text{with }(k,l)\in K^2\text{ and }a=a(g)\in A^+, \tag{2.5}$$

in which $a(g)\in A^+$ is unique and called the *Cartan projection* of g.

2.2.2. *Asymptotic directions in* $\mathrm{PGL}_3(\mathbb{R})$. The standard decomposition of elements of $\mathrm{PGL}_3(\mathbb{R})$ allows us, with the help of relation (2.3), to reduce the investigation of dynamic sets of sequences of $\mathrm{PGL}_3(\mathbb{R})$ to the specific case of sequences of A^+. Therefore, we now focus on sequences of A^+ going to infinity.

Definition 2.3. A sequence $a_n = \mathrm{Diag}(\alpha_n, \beta_n, \gamma_n) \in A^+$ goes *simply to infinity* if it goes to infinity with the three sequences $\frac{\alpha_n}{\beta_n} \geq 1$, $\frac{\alpha_n}{\gamma_n} \geq 1$, $\frac{\beta_n}{\gamma_n} \geq 1$ having a limit in $[1\,;+\infty]$. A sequence $g_n \in \mathrm{PGL}_3(\mathbb{R})$ goes *simply to infinity* if there exists a standard decomposition $g_n = k_n a_n l_n$ whose factors k_n and l_n in K converge and whose Cartan projection $a_n = a(g_n) \in A^+$ goes simply to infinity.

It is easy to check that for any sequence $a_n = \mathrm{Diag}(\alpha_n, \beta_n, \gamma_n) \in A^+$ going simply to infinity, we have $\lim \frac{\alpha_n}{\gamma_n} = +\infty$. The dynamics of (a_n) does thus only depend on the ratios $\lim \frac{\alpha_n}{\beta_n}$ and $\lim \frac{\beta_n}{\gamma_n}$, which ends up in three distinct *asymptotic types* in $\mathrm{PGL}_3(\mathbb{R})$.

Definition 2.4. Let (g_n) be a sequence of $\mathrm{PGL}_3(\mathbb{R})$ going simply to infinity, and $\mathrm{Diag}(\alpha_n, \beta_n, \gamma_n) \in A^+$ be its Cartan projection.

– If $\lim \frac{\alpha_n}{\beta_n} = +\infty$ and $\lim \frac{\beta_n}{\gamma_n} < +\infty$, we will say that (g_n) is of *unbalanced type* α.
– If $\lim \frac{\alpha_n}{\beta_n} < +\infty$ et $\lim \frac{\beta_n}{\gamma_n} = +\infty$, we will say that (g_n) is of *unbalanced type* β.
– If $\lim \frac{\alpha_n}{\beta_n} = \lim \frac{\beta_n}{\gamma_n} = +\infty$, we will say that (g_n) is of *balanced type*.

Remark 2.5. In the general setting of Anosov representations, these three types of sequences are called θ*-divergent* or P_θ*-divergent* for suitable subsets θ of the (fixed) set $\Delta \subset \mathfrak{a}^*$ of simple restricted roots of the Lie algebra of the target group (P_θ being the associated parabolic subgroup). In our case, the Cartan subspace $\mathfrak{a}$ is the set of diagonal matrices of $\mathfrak{sl}_3(\mathbb{R})$, and we choose $\Delta = \{l_1 = e_1^* - e_2^*, l_2 = e_2^* - e_3^*\}$ where $e_i^*(\mathrm{Diag}(x_1, x_2, x_3)) = x_i$, and define $\mu \colon \mathrm{PGL}_3(\mathbb{R}) \to \mathfrak{a}$ by $a(g) = \exp(\mu(g))$. Then for $\theta \subset \Delta$, a sequence $g_n \in \mathrm{PGL}_3(\mathbb{R})$ is said to be θ*-divergent* if for any $\alpha \in \theta$: $\lim \alpha(\mu(g_n)) = +\infty$. This notion appears with several variations in the literature of Anosov representations, being sometime formulated for the whole discrete group being represented and called α_i*-divergent* group in [GW12, Definition 7.2] or P_θ*-divergent* representation in [GGKW17, p.539] (see also the related notion of τ_{mod}*-regular sequence* [KLP17, Definition 4.4] formulated in terms of a face τ_{mod} of a model spherical Weyl chamber). The correspondence with Definition 2.4 is then:

– (g_n) is of unbalanced type α if it is l_1-divergent and not l_2-divergent;
– (g_n) is of unbalanced type β if it is l_2-divergent and not l_1-divergent;
– (g_n) is of balanced type if it is $\{l_1, l_2\}$-divergent (*i.e.* $\mathbf{P}_{min}$-divergent).

Remark 2.6. By compacity of K, any sequence (g_n) going to infinity in $\mathrm{PGL}_3(\mathbb{R})$ has a subsequence going simply to infinity. Furthermore, the possible asymptotic types of the subsequences of (g_n) going simply to infinity do only depend on those of the subsequences of its Cartan projection $(a(g_n))$. In particular, all the subsequences of a sequence (g_n) going simply to infinity have the same asymptotic type than (g_n). Furthermore, one easily checks that if (g_n) is a sequence going simply to infinity in $\mathrm{PGL}_3(\mathbb{R})$ and (k_n), (l_n) are relatively compact sequences in $\mathrm{PGL}_3(\mathbb{R})$, then any subsequence of $(k_n g_n l_n)$ going simply to infinity has the same asymptotic type than (g_n).

Among sequences of $\mathrm{PGL}_3(\mathbb{R})$, iterates of a fixed element of $\mathrm{PGL}_3(\mathbb{R})$ are particularly important examples. We will say (by a slight misuse of language) that $g \in \mathrm{PGL}_3(\mathbb{R})$ is *diagonalizable* if it has a representative $g_0 \in \mathrm{GL}_3(\mathbb{R})$ which is diagonalizable *on* $\mathbb{R}$. The following claim is then a straightforward application of Definition 2.4.

Lemma 2.7. *Let* $g \in \mathrm{PGL}_3(\mathbb{R})$ *be a diagonalizable element. Then:*

– (g^n) *goes to infinity in* $\mathrm{PGL}_3(\mathbb{R})$ *if, and only if we do not have* $a = b = c$ *with* $a \geq b \geq c > 0$ *the absolute values of the eigenvalues of* g *counted with multiplicity;*
– *if* (g^n) *goes to infinity, then it goes simply to infinity if, and only if its three eigenvalues have the same sign.*

Furthermore, if (g^n) *goes to infinity, then its subsequences going simply to infinity all have the same asymptotical type:*

- *unbalanced type* α *if* $a > b = c$*;*
- *unbalanced type* β *if* $a = b > c$*;*
- *balanced type if* $a > b > c$*, in which case we will say that* g *is* loxodromic*.*

We conclude the description of asymptotic directions in $\mathrm{PGL}_3(\mathbb{R})$ with the following duality.

Lemma 2.8. *Let* (g_n) *be a sequence of* $\mathrm{PGL}_3(\mathbb{R})$ *going simply to infinity. Then* (g_n^{-1}) *goes simply to infinity, and if* (g_n) *is of unbalanced type* α *(respectively unbalanced type* β*, resp. balanced type), then* (g_n^{-1}) *is of unbalanced type* β *(resp. unbalanced type* α*, resp. balanced type).*

Proof. Thanks to standard decomposition 2.5 and relation 2.3, it is sufficient to prove this for a sequence $a_n = \mathrm{Diag}(\alpha_n, \beta_n, \gamma_n) \in A^+$ going simply to infinity, and we moreover assume (a_n) of unbalanced type α, the argument being similar in the two other cases. Then $a_n^* \coloneqq \mathrm{Diag}(\gamma_n^{-1}, \beta_n^{-1}, \alpha_n^{-1}) \in A^+$ goes simply to infinity with unbalanced type β, and with

$$I = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} = I^{-1} \in K \tag{2.6}$$

we have $a_n^{-1} = I a_n^* I^{-1}$, showing that (a_n^{-1}) is also of unbalanced type β. □

2.3. **Dynamics in the projective plane.** In this section we give a systematic description of the dynamic sets of points of $\mathbb{R}\mathbf{P}^2$ for the action of sequences of $\mathrm{PGL}_3(\mathbb{R})$ escaping to infinity, depending on the three possible asymptotic behaviours previously described. Some of these results are likely to be already known (see for instance [Gol87, §3.3] for related material), but we give here precise statements and complete proofs for the convenience of the reader.

For $p \in \mathbb{R}\mathbf{P}^2$ we define the *dual projective line of* p as $p^* = \{D \in \mathbb{R}\mathbf{P}^2_* \mid D \ni p\}$. We also recall that for $P \subset \mathbb{R}^3$, $[P]$ denotes the projection of $\mathrm{Vect}\, P$ in $\mathbb{R}\mathbf{P}^2$, and that (e_1, e_2, e_3) denotes the standard basis of $\mathbb{R}^3$.

Lemma 2.9. *Let* $g_n \in \mathrm{PGL}_3(\mathbb{R})$ *be a sequence going simply to infinity with unbalanced type* α*. Then there exists a projective line* D_- *and a point* p_+ *in* $\mathbb{R}\mathbf{P}^2$*, respectively called the* repulsive line *and the* attractive point *of* (g_n)*, as well as a diffeomorphism* $\hat{g}_\infty \colon D_- \to (p_+)^*$*, satisfying the following.*

1. *For any* $p \in \mathbb{R}\mathbf{P}^2 \setminus D_-$*,* $\mathcal{D}_{(g_n)}(p) = p_+$*.*
2. *For any* $p \in D_-$*,* $\mathcal{D}_{(g_n)}(p) = \hat{g}_\infty(p)$*.*
3. $\hat{g}_\infty$ *is equivariant for a morphism* $\rho_\infty \colon \mathrm{Stab}(D_-) \to \mathrm{Stab}(D_-) \cap \mathrm{Stab}(p_+)$*.*

If moreover $g_n \in A^+$*, then* $p_+ = [e_1]$ *and* $D_- = [e_2, e_3]$*.*

Proof. Let us assume that $g_n = k_n a_n l_n$ is a standard decomposition of g_n as defined in (2.5), with $\lim k_n = k_\infty$, $\lim l_n = l_\infty$ and (a_n) going simply to infinity with unbalanced type α. Then if we assume that we already proved the Lemma for sequences of A^+, it is easy to check, using the relation (2.3) between dynamic sets, that (g_n) verifies the claims of the Lemma with $D_-(g_n) = l_\infty^{-1}(D_-(a_n))$, $p_+(g_n) = k_\infty(p_+(a_n))$ and $\hat{g}_\infty = k_\infty \circ \hat{a}_\infty \circ l_\infty$. We thus only have to prove the Lemma for sequences

$$a_n = \begin{bmatrix} 1 & & \\ & \beta_n & \\ & & \gamma_n \end{bmatrix} \in A^+,$$

going simply to infinity with unbalanced type α. Therefore $\lim \beta_n = \lim \gamma_n = 0$ and $\lim \frac{\gamma_n}{\beta_n} = \lambda_\infty \in \,]0\,;1]$. We define $p_+ = [e_1]$ and $D_- = [e_2, e_3]$.

1. Since $\mathbb{R}\mathbf{P}^2$ is compact, $\mathcal{D}_{(a_n)}(p)$ is non-empty and it is sufficient to show the direct inclusion. If $p_n \in \mathbb{R}\mathbf{P}^2$ converges to $[e_1]$, then for n large enough there exists $(x_n, y_n) \in \mathbb{R}^2$ converging to $(0,0)$ such that $p_n = [1 : x_n : y_n]$. Hence $a_n(p_n) = [1 : \beta_n x_n : \gamma_n y_n]$ converges to $[e_1]$, showing that $\mathcal{D}_{(a_n)}(p) \subset [e_1]$ as claimed.

2. We define $a_\infty = \mathrm{Diag}(1, 1, \lambda_\infty)$ and $\hat{a}_\infty(p) = [p_+, a_\infty(p)]$. For convenience in the notations, we assume that for some $y \in \mathbb{R}$, $p = [0 : 1 : y] \in [e_2, e_3] \setminus \{[e_3]\}$ (the proof being similar if $p \neq [e_2]$).

We first show that $\mathcal{D}_{(a_n)}(p) \subset \hat{a}_\infty(p)$. Since $a_n|_{[e_2,e_3]}$ uniformly converges to $a_\infty|_{[e_2,e_3]}$, for any sequence $p_n \in [e_2, e_3]$ converging to p we readily have $\lim a_n(p_n) = a_\infty(p) \in \hat{a}_\infty(p)$. We thus assume that $p_n = [1 : x_n : y_n] \in \mathbb{R}\mathbf{P}^2 \setminus [e_2, e_3]$ converges to p, implying $\lim \frac{1}{x_n} = 0$ and $\lim \frac{y_n}{x_n} = y$. Passing to a subsequence, we can assume that $a_n(p_n) = [1 : \beta_n x_n : \gamma_n y_n]$ converges to $q \in \mathbb{R}\mathbf{P}^2$ and we want to prove that $q \in \hat{a}_\infty(p)$. If $q \in [e_2, e_3]$ then $\lim \|(\beta_n x_n, \gamma_n y_n)\| = +\infty$, and since $\lim \left|\frac{\gamma_n y_n}{\beta_n x_n}\right| = \lambda_\infty y < +\infty$ this prevents $(\beta_n x_n)$ to be bounded. Passing to a subsequence we thus have $\lim |\beta_n x_n| = +\infty$, and $q = \lim[\frac{1}{\beta_n x_n} : 1 : \frac{\gamma_n y_n}{\beta_n x_n}] = [0 : 1 : \lambda_\infty y] = a_\infty(p) \in \hat{a}_\infty(p)$. If $q = [1 : x_\infty : y_\infty] \in \mathbb{R}\mathbf{P}^2 \setminus [e_2, e_3]$ then $\frac{y_\infty}{x_\infty} = \lim \frac{\gamma_n y_n}{\beta_n x_n} = \lambda_\infty y$, which exactly means that $q \in [e_1, (0, 1, \lambda_\infty y)] = \hat{a}_\infty(p)$.

We now show the reverse inclusion $\hat{a}_\infty(p) \subset \mathcal{D}_{(a_n)}(p)$. For $t \in \mathbb{R}^*$, the sequence $p_n := [1 : \frac{t}{\beta_n} : \frac{yt}{\beta_n}] = [\frac{\beta_n}{t} : 1 : y]$ converges to p while $a_n(p_n)$ converges to $q = [1 : t : \lambda_\infty y t] \in [e_1, a_\infty(p)] \cap (\mathbb{R}\mathbf{P}^2 \setminus [e_2, e_3])$. This shows that $\hat{a}_\infty(p) \setminus \{e_1, a_\infty(p)\} \subset \mathcal{D}_{(a_n)}(p)$ which implies $\hat{a}_\infty(p) \subset \mathcal{D}_{(a_n)}(p)$ since the latter is closed.

3. Let us denote by $b_\infty = \mathrm{Diag}(1, \lambda_\infty) \in \mathrm{PGL}_2(\mathbb{R})$ the restriction of a_∞ to D^-. Then $\hat{a}_\infty$ is equivariant for the following morphism:

$$\rho_\infty \colon \begin{pmatrix} 1 & 0 \\ * & g \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & b_\infty g b_\infty^{-1} \end{pmatrix}. \qquad \square$$

Lemma 2.10. *Let $g_n \in \mathrm{PGL}_3(\mathbb{R})$ be a sequence going simply to infinity with unbalanced type β. Then there exists a point p_- and a projective line D_+ in $\mathbb{R}\mathbf{P}^2$, respectively called the* repulsive point *and the* attractive line *of (g_n), as well as a $\mathbb{R}$-bundle $\bar{g}_\infty \colon \mathbb{R}\mathbf{P}^2 \setminus \{p_-\} \to D_+$, satisfying the following.*

1. *For any $p \in \mathbb{R}\mathbf{P}^2 \setminus \{p_-\}$, $\mathcal{D}_{(g_n)}(p) = \bar{g}_\infty(p)$.*
2. *$\mathcal{D}_{(g_n)}(p_-) = \mathbb{R}\mathbf{P}^2$.*
3. *$\bar{g}_\infty$ is equivariant for a morphism $\rho_\infty \colon \mathrm{Stab}(p_-) \to \mathrm{Stab}(D_+) \cap \mathrm{Stab}(p_-)$.*

If moreover $g_n \in A^+$, then $p_- = [e_3]$ and $D_+ = [e_1, e_2]$.

Proof. As we saw in the proof of Lemma 2.9, we only have to prove the claims for a sequence

$$a_n = \begin{bmatrix} \alpha_n & & \\ & \beta_n & \\ & & 1 \end{bmatrix} \in A^+$$

going simply to infinity with unbalanced type β, therefore $\lim \alpha_n = \lim \beta_n = +\infty$ and $\lim \frac{\beta_n}{\alpha_n} = \lambda_\infty \in \,]0\,;1]$. We define $p_- = e_3$ and $D_+ = [e_1, e_2]$.

1. With $a_\infty = \mathrm{Diag}(1, \lambda_\infty, 1)$, the fibration is $\bar{a}_\infty(p) = a_\infty([p_-, p] \cap D_+)$. According to Lemma 2.8 (a_n^{-1}) goes simply to infinity with unbalanced type α, and since $a_n^{-1} = g_0 b_n g_0^{-1}$ with $b_n = \mathrm{Diag}(1, \beta_n^{-1}, \alpha_n^{-1}) \in A^+$ and $g_0 = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$, the attractive point and repulsive line of (a_n^{-1}) are respectively $[e_3]$ and $[e_1, e_2]$ and $\widehat{a^{-1}}_\infty(p) = [e_3, a_\infty^{-1}(p)]$ for any $p \in [e_1, e_2]$. With $p \neq p_-$, we only have to prove that $\mathcal{D}_{(a_n)}(p) \subset \{\bar{a}_\infty(p)\}$ since $\mathcal{D}_{(a_n)}(p)$ is non-empty. We take $q \in \mathcal{D}_{(a_n)}$ and passing to a subsequence of (a_n) we can assume that $q = \lim a_n(p_n)$ with $p_n \in \mathbb{R}\mathbf{P}^2$ such that $\lim p_n = p$. Denoting $q_n = a_n(p_n)$, since $\lim a_n^{-1}(q_n) = p \neq [e_3]$ is not the attractive point of (a_n^{-1}), $q = \lim q_n$ belong to its repulsive line $[e_1, e_2]$ according to Lemma 2.9. We thus have $p \in [e_3, a_\infty^{-1}(q)]$ and thus $a_\infty^{-1}(q) \in [e_3, p]$. Therefore $q = [e_3, a_\infty(p)] \cap [e_1, e_2] = \bar{a}_\infty(p)$ as claimed.

2. For any $q = [x : y : 1] \in \mathbb{R}\mathbf{P}^2 \setminus D_+$, $p_n := [\frac{x}{\alpha_n} : \frac{y}{\beta_n} : 1]$ converges to p_- and $a_n(p_n) = q$ converges to q. Hence $\mathbb{R}\mathbf{P}^2 \setminus D_+ \subset \mathcal{D}_{(a_n)}(p_-)$, showing the claim since $\mathcal{D}_{(a_n)}(p_-)$ is closed.

3. Denoting $b_\infty = \mathrm{Diag}(1, \lambda_\infty) \in \mathrm{PGL}_2(\mathbb{R})$, $\bar{a}_\infty$ is equivariant for the following morphism:

$$\rho_\infty \colon \begin{pmatrix} g & 0 \\ * & 1 \end{pmatrix} \mapsto \begin{pmatrix} b_\infty g b_\infty^{-1} & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.7}$$

$\square$

We will say that two flags (p, D) and (p', D') in $\mathbf{X}$ are *in general position* if $p \notin D'$ and $p' \notin D$.

Lemma 2.11. *Let $g_n \in \mathrm{PGL}_3(\mathbb{R})$ be a sequence going simply to infinity with balanced type. Then there exists two projective lines D_-, D_+ and two points $p_- \in D_-$, $p_+ \in D_+$ of $\mathbb{R}\mathbf{P}^2$, respectively called the* repulsive and attractive lines *and* points *of (g_n), satisfying the following.*

1. *For any $p \in \mathbb{R}\mathbf{P}^2 \setminus D_-$, $\mathcal{D}_{(g_n)}(p) = \{p_+\}$.*
2. *For any $p \in D_- \setminus \{p_-\}$, $\mathcal{D}_{(g_n)}(p) = D_+$.*
3. *$\mathcal{D}_{(g_n)}(p_-) = \mathbb{R}\mathbf{P}^2$.*

If $x_- = (p_-, D_-)$ and $x_+ = (p_+, D_+)$ are in general position then $p_\pm \coloneqq D_- \cap D_+ \in \mathbb{R}\mathbf{P}^2$ is called the saddle-point *of (g_n). Moreover if $g_n \in A^+$, then $p_+ = [e_1]$, $p_\pm = [e_2]$ and $p_- = [e_3]$.*

Proof. As we saw in the proof of Lemma 2.9, we only have to prove the claims for a sequence

$$a_n = \begin{bmatrix} 1 & & \\ & \beta_n & \\ & & \gamma_n \end{bmatrix} \in A^+$$

going simply to infinity with balanced type, therefore $\lim \beta_n = \lim \gamma_n = \lim \frac{\gamma_n}{\beta_n} = 0$. We define $p_- = [e_3] \in D_- = [e_2, e_3]$, $p_+ = [e_1] \in D_+ = [e_1, e_2]$ and $p_\pm = [e_2]$.
1. The proof of this first claim is very similar to the one of the first claim of Lemma 2.9.
2. Let $p = [0 : 1 : y] \in D_- \setminus \{p_-\}$ with $y \in \mathbb{R}$, and let $p_n \in \mathbb{R}\mathbf{P}^2$ be a sequence converging to p. Passing to a subsequence we can assume that $a_n(p_n)$ converges to a point $p \in \mathbb{R}\mathbf{P}^2$ and we now show that $q \in D_+$. If some subsequence of (p_n) is contained in D_-, then for n large enough $p_n = [0 : 1 : y_n]$ for some y_n converging to y and $a_n(p_n) = [0 : 1 : \frac{\gamma_n}{\beta_n} y_n]$ converges thus to $e_2 = p_\pm$. If not, there exists a sequence $(x_n, y_n) \in \mathbb{R}^2$ such that $p_n = [1 : x_n : y_n]$ for n large enough, and we thus have $\lim \frac{1}{x_n} = 0$ and $\lim \frac{y_n}{x_n} = y$ since (p_n) converges to p. We first assume that $a_n(p_n) = [1 : \beta_n x_n : \gamma_n y_n]$ converges to $q \in [e_2, e_3]$. If $(\beta_n x_n)$ was bounded then $\lim \gamma_n y_n = 0$ since $\lim \frac{\gamma_n y_n}{\beta_n x_n} = 0$, and thus $(\beta_n x_n, \gamma_n y_n)$ would be bounded which contradicts $\lim [1 : \beta_n x_n : \gamma_n y_n] \in [e_2, e_3]$. Passing to a subsequence, we thus have $\lim |\beta_n x_n| = +\infty$ and $q = [e_2] \in D_+$. We now assume that $q = [1 : x_\infty : y_\infty] \notin [e_2, e_3]$. Then $y_\infty = 0$ since $\lim \frac{\gamma_n y_n}{\beta_n x_n} = 0$, hence $q \in D_+$ again which concludes the proof of the inclusion $\mathcal{D}_{(a_n)}(p) \subset D_+$.

Conversely for any $t \in \mathbb{R}^*$, $p_n = [1 : \frac{t}{\beta_n} : \frac{ty}{\beta_n}]$ converges to $p = [0 : 1 : y]$ while $a_n(p_n) = [1 : t : ty\frac{\gamma_n}{\beta_n}]$ converges to $[1 : t : 0]$. This shows that $D_+ \setminus \{p_+, p_-\} \subset \mathcal{D}_{(a_n)}(p)$, hence $D_+ \subset \mathcal{D}_{(a_n)}(p)$ since $\mathcal{D}_{(a_n)}(p)$ is closed.
3. For any $(x, y) \in \mathbb{R}^2$ with $y \neq 0$, $p_n = [1 : \frac{x}{\beta_n} : \frac{y}{\gamma_n}]$ converges to p_- while $a_n(p_n)$ converges to $[1 : x : y]$. This shows that $\mathbb{R}\mathbf{P}^2 \setminus ([e_2, e_3] \cup [e_1, e_2]) \subset \mathcal{D}_{(a_n)}(p_-)$, hence $\mathbb{R}\mathbf{P}^2 = \mathcal{D}_{(a_n)}(p_-)$ since the latter is closed. □

Example 2.12. Let $g \in \mathrm{PGL}_3(\mathbb{R})$ be a loxodromic element. We already know from Lemma 2.7 that (g^n) goes to infinity, and that any subsequence of (g^n) going simply to infinity has balanced type. The proof of Lemma 2.11 furthermore learns us that the repulsive, saddle and attractive points of any such subsequence are the three eigenlines p_-, $p_\pm$ and p_+ of a representative of g, arranged in the ascending order of the absolute value of their associated eigenvalues.

Remark 2.13. We saw in the proof of the three previous Lemmas that for sequences of A^+, the dynamical objects are disjoints ($p_+ \notin D_-$ for the unbalanced type α, $p_- \notin D_+$ for the unbalanced type β, (p_-, D_-) and (p_+, D_+) are in general position for the balanced type). Note however that in general, there is no reason to expect the dynamical objects to be disjoints. For instance, one can check that with

$$g = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

any subsequence of (g^n) going simply to infinity has balanced type, with dynamical objects $p_- = p_+ = [e_1]$ and $D_- = D_+ = [e_1, e_3]$.

Remark 2.14. We saw in Lemma 2.8 the following duality of asymptotic directions in $\mathrm{PGL}_3(\mathbb{R})$: if $g_n \in \mathrm{PGL}_3(\mathbb{R})$ goes simply to infinity with unbalanced type α (respectively β, respectively balanced type), then (g_n^{-1}) goes simply to infinity with unbalanced type β (resp. α, resp. balanced type). As one could expect, this duality also applies in the naive way to dynamical objects: repulsive objects of (g_n^{-1}) are the corresponding attractive objects of (g_n). For instance if (g_n) has balanced type, then $p_-(g_n^{-1}) = p_+(g_n)$ and $D_-(g_n^{-1}) = D_+(g_n)$, and conversely. These relations are easily verified for a sequence $a_n \in A^+$ which readily implies the general case by using standard decomposition (2.5) and relation (2.3) between dynamic sets.

2.4. **Dynamics in the dual projective plane.** Denoting by $P^\perp$ the orthogonal subspace of $P \subset \mathbb{R}^3$ for the standard euclidean scalar product, the diffeomorphism

$$\tau\colon m \in \mathbb{R}\mathbf{P}^2 \mapsto [m^\perp] \in \mathbb{R}\mathbf{P}^2_* \tag{2.8}$$

between the projective space and its dual is equivariant with respect to the involutive morphism

$$\Theta\colon g \in \mathrm{PGL}_3(\mathbb{R}) \mapsto {}^t g^{-1} \in \mathrm{PGL}_3(\mathbb{R}). \tag{2.9}$$

The dynamics on $\mathbb{R}\mathbf{P}^2_*$ of a sequence (g_n) of unbalanced type α (respectively unbalanced type β, resp. balanced type) will thus be conjugated by τ to the dynamics on $\mathbb{R}\mathbf{P}^2$ of the sequence $({}^t g_n{}^{-1})$ of unbalanced type β (resp. unbalanced type α, resp. balanced type). Furthermore, dynamical objects of (g_n) acting on $\mathbb{R}\mathbf{P}^2_*$ are directly deduced from its dynamical objects in $\mathbb{R}\mathbf{P}^2$ as described in the following Lemmas.

Lemma 2.15. *Let $g_n \in \mathrm{PGL}_3(\mathbb{R})$ going simply to infinity of unbalanced type α, with repulsive projective line D_- and attractive point p_+ in $\mathbb{R}\mathbf{P}^2$. Denoting by $\hat{g}_\infty\colon D_- \to p_+^*$ the diffeomorphism of Lemma 2.9, (g_n) has on $\mathbb{R}\mathbf{P}^2_*$ the following dynamics:*

1. *for $D \in \mathbb{R}\mathbf{P}^2_* \setminus \{D_-\}$, $\mathcal{D}_{(g_n)}(D) = \hat{g}_\infty(D \cap D_-) \in (p_+)^*$;*
2. *$\mathcal{D}_{(g_n)}(D_-) = \mathbb{R}\mathbf{P}^2_*$.*

There exists of course a corresponding result for sequences of unbalanced type β, although we will not state it here as it will not be used in this paper.

Lemma 2.16. *Let $g_n \in \mathrm{PGL}_3(\mathbb{R})$ going simply to infinity of balanced type, with repulsive and attractive points and projective lines $p_- \in D_-$ and $p_+ \in D_+$ in $\mathbb{R}\mathbf{P}^2$. Then (g_n) has on $\mathbb{R}\mathbf{P}^2_*$ the following dynamics:*

1. *for $D \in \mathbb{R}\mathbf{P}^2_* \setminus (p_-)^*$, $\mathcal{D}_{(g_n)}(D) = D_+$;*
2. *for $D \in (p_-)^* \setminus \{D_-\}$, $\mathcal{D}_{(g_n)}(D) = (p_+)^*$;*
3. *$\mathcal{D}_{(g_n)}(D_-) = \mathbb{R}\mathbf{P}^2_*$.*

As before, these results are easily proved in the case of sequences of A^+, which yields the general case by using standard decomposition (2.5).

2.5. **Dynamics in the flag space.** From our description of the dynamics of $\mathrm{PGL}_3(\mathbb{R})$ on $\mathbb{R}\mathbf{P}^2$ and $\mathbb{R}\mathbf{P}^2_*$, we now deduce a precise description of its dynamics on the flag space $\mathbf{X}$ (see (2.1)). We refer to [KLP17, §6] for results related to those of this paragraph, in the general setting of *regular* discrete subgroups of semi-simple Lie groups.

2.5.1. *Geometry of the flag space.* The dynamical repulsive and attractive objects of a sequence of $\mathrm{PGL}_3(\mathbb{R})$ acting on $\mathbf{X}$ are natural geometric objects of $\mathbf{X}$ that we now define. First of all, $\mathbf{X}$ bears a path structure $\mathcal{L}_{\mathbf{X}} = (\mathcal{E}_\alpha, \mathcal{E}_\beta)$ whose associated one-dimensional α and β-leaves are the respective fibers of the two following projections:

$$\pi_\alpha\colon (p, D) \in \mathbf{X} \mapsto p \in \mathbb{R}\mathbf{P}^2 \text{ and } \pi_\beta\colon (p, D) \in \mathbf{X} \mapsto D \in \mathbb{R}\mathbf{P}^2_*. \tag{2.10}$$

In other words, $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ are respectively tangent at $x = (p, D) \in \mathbf{X}$ to the circles

$$\mathcal{C}_\alpha(x) = \{(p, D') \mid D' \ni p\} = \pi_\alpha^{-1}(p) \text{ and } \mathcal{C}_\beta(x) = \{(p', D) \mid p' \in D\} = \pi_\beta^{-1}(D) \tag{2.11}$$

that we respectively call the *α-circle* and the *β-circle* of x. We will also denote $\mathcal{C}_\alpha(x) = \mathcal{C}_\alpha(p)$ and $\mathcal{C}_\beta(x) = \mathcal{C}_\beta(D)$. The fact that $\mathcal{E}_\alpha \oplus \mathcal{E}_\beta$ is a contact distribution and that $\mathcal{L}_{\mathbf{X}}$ is thus a path structure on $\mathbf{X}$ is classical and can be verified by a calculation in a chart of $\mathbf{X}$. Moreover

π_α and π_β are $\mathrm{PGL}_3(\mathbb{R})$-equivariant for the natural action of $\mathrm{PGL}_3(\mathbb{R})$ on $\mathbf{X}$ and $\mathcal{L}_{\mathbf{X}}$ is thus $\mathrm{PGL}_3(\mathbb{R})$-invariant. Actually, $\mathrm{PGL}_3(\mathbb{R})$ is the whole automorphism group of $(\mathbf{X}, \mathcal{L}_{\mathbf{X}})$ (see for instance [MM21, Lemma 2.2]), where an *automorphism* of a path structure $\mathcal{L} = (E^\alpha, E^\beta)$ on a manifold M is a diffeomorphism f of M such that $f^*E^\alpha = E^\alpha$ and $f^*E^\beta = E^\beta$.

One obtains natural surfaces of $\mathbf{X}$ by defining

$$\mathcal{S}_{\alpha,\beta}(x) = \bigcup_{y\in\mathcal{C}_\alpha(x)} \mathcal{C}_\beta(y) \text{ and } \mathcal{S}_{\beta,\alpha}(x) = \bigcup_{y\in\mathcal{C}_\beta(x)} \mathcal{C}_\alpha(y) \tag{2.12}$$

that we respectively call the α-β and the β-α *surfaces* of $x = (p, D)$. As before, we will also denote $\mathcal{S}_{\alpha,\beta}(x) = \mathcal{S}_{\alpha,\beta}(p) = \pi_\beta^{-1}(p^*)$ and $\mathcal{S}_{\beta,\alpha}(x) = \mathcal{S}_{\beta,\alpha}(D) = \pi_\alpha^{-1}(D)$ if $x = (p, D)$. Note that α-β and β-α surfaces are compact and connected surfaces of Euler characteristic zero according to Poincaré-Hopf Theorem (because each of these surfaces bears a one-dimensional distribution which is the restriction of $\mathcal{E}_\beta$ or $\mathcal{E}_\alpha$). One moreover checks that these surfaces are non-orientable, and that α-β and β-α surfaces are thus Klein bottles embedded in $\mathbf{X}$.

To finish this short geometric introduction to the flag space, we introduce the following natural involutive diffeomorphism of $\mathbf{X}$:

$$\kappa\colon (m, D) \in \mathbf{X} \mapsto (D^\perp, m^\perp) \in \mathbf{X}, \tag{2.13}$$

called the *dual involution of* $\mathbf{X}$, which is equivariant for the automorphism $\Theta\colon g \mapsto {}^t g^{-1}$ of $\mathrm{PGL}_3(\mathbb{R})$. Note that κ does not preserve $\mathcal{L}_{\mathbf{X}}$ but switches its α and β-distributions: $\kappa^*\mathcal{E}_\alpha = \mathcal{E}_\beta$. From this, one easily verifies the relations

$$\kappa(\mathcal{C}_\alpha(x)) = \mathcal{C}_\beta(\kappa(x)), \kappa(\mathcal{C}_\beta(x)) = \mathcal{C}_\alpha(\kappa(x)), \kappa(\mathcal{S}_{\alpha,\beta}(x)) = \mathcal{S}_{\beta,\alpha}(\kappa(x)) \text{ and } \kappa(\mathcal{S}_{\beta,\alpha}(x)) = \mathcal{S}_{\alpha,\beta}(\kappa(x)).$$

2.5.2. *Unbalanced type α.* For $g_n \in \mathrm{PGL}_3(\mathbb{R})$ a sequence going simply to infinity of unbalanced type α, with repulsive projective line D_- and attractive point p_+ in $\mathbb{R}\mathbf{P}^2$, we define $\mathcal{C}_\beta^- = \mathcal{C}_\beta(D_-)$, $\mathcal{S}_{\beta,\alpha}^- = \mathcal{S}_{\beta,\alpha}(D_-)$, $\mathcal{C}_\alpha^+ = \mathcal{C}_\alpha(p_+)$ and $\mathcal{S}_{\alpha,\beta}^+ = \mathcal{S}_{\alpha,\beta}(p_+)$.

Lemma 2.17. *There exists a surjective application $\phi\colon \mathbf{X} \to \mathcal{C}_\alpha^+$ such that:*

1. *$\phi|_{\mathbf{X}\setminus\mathcal{C}_\beta^-}$ is a (smooth) $(\mathbf{S}^1 \times \mathbb{R})$-fiber bundle whose fibers are the $\mathcal{S}_{\alpha,\beta}(x) \setminus \mathcal{C}_\beta^-$ for $x \in \mathcal{C}_\beta^-$, but ϕ is not continuous on $\mathcal{C}_\beta^-$;*
2. *for $x \in \mathbf{X} \setminus \mathcal{S}_{\beta,\alpha}^-$, $\mathcal{D}_{(g_n)}(x) = \phi(x)$;*
3. *for $x \in \mathcal{S}_{\beta,\alpha}^- \setminus \mathcal{C}_\beta^-$, $\mathcal{D}_{(g_n)}(x) = \mathcal{C}_\beta(\phi(x))$;*
4. *for $x \in \mathcal{C}_\beta^-$, $\mathcal{D}_{(g_n)}(x) = \mathcal{S}_{\beta,\alpha}(\phi(x))$;*
5. *ϕ is equivariant for the morphism $\rho_\infty\colon \mathrm{Stab}(D_-) \to \mathrm{Stab}(p_+) \cap \mathrm{Stab}(D_-)$ of Lemma 2.9.*

Proof. Denoting $x = (p, D)$, we define $\phi(x) = (p_+, \hat{g}_\infty(D \cap D_-))$ if $x \notin \mathcal{C}_\beta^-$ and $\phi(x) = (p_+, \hat{g}_\infty(p))$ if $x \in \mathcal{C}_\beta^-$, with $\hat{g}_\infty$ the application introduced in Lemma 2.9.
1. These claims are immediate consequences of the definition of ϕ.
2. If $x \notin \mathcal{S}_{\beta,\alpha}^-$ then $p \notin D_-$ hence $\mathcal{D}_{(g_n)}(p) = \{p_+\}$ according to Lemma 2.9. Moreover $D \neq D_-$, hence $\mathcal{D}_{(g_n)}(D) = \{\hat{g}_\infty(D \cap D_-)\}$ according to Lemma 2.15. This proves the claim.
3. If $x \notin \mathcal{C}_\beta^-$ then $D \neq D_-$ and thus $\mathcal{D}_{(g_n)}(D) = \{\hat{g}_\infty(D \cap D_-)\}$, proving the direct inclusion. For the reverse inclusion, let $p_\infty \in \hat{g}_\infty(D \cap D_-)$. Since $D \cap D_- = p \in D_-$, according to Lemma 2.9 there exists $p_n \in \mathbb{R}\mathbf{P}^2$ converging to p such that $g_n(p_n)$ converges to p_∞. For n large enough $p_n \notin p^\perp$ and thus $D_n = [p_n, p^\perp \cap D]$ is a projective line converging to D. Hence $x_n = (p_n, D_n)$ converges to x, with $g_n(x_n)$ converging to $(p_\infty, \hat{g}_\infty(D \cap D_-))$. This proves the equality.
4. If $x = (p, D_-) \in \mathcal{C}_\beta^-$ then $\mathcal{D}_{(g_n)}(p) = \hat{g}_\infty(p)$ according to Lemma 2.9, proving the direct inclusion. For the reverse one, let $x_\infty = (p_\infty, D_\infty) \in \mathcal{S}_{\beta,\alpha}(\hat{g}_\infty(p)) \setminus \mathcal{C}_\beta(\hat{g}_\infty(p))$, that is $p_\infty \in \hat{g}_\infty(p)$ and $D_\infty \neq \hat{g}_\infty(p)$, hence $D_\infty \cap \hat{g}_\infty(p) = \{p_\infty\}$. Lemma 2.9 gives a sequence $p_n \in \mathbb{R}\mathbf{P}^2$ converging to p such that $g_n(p_n)$ converges to p_∞. With $q \in D_\infty \setminus \{p_+, p_\infty\}$, for n large enough $[g_n(p_n), q]$ is a projective live converging to D_∞. According to Remark 2.14, (g_n^{-1}) is a sequence of unbalanced type β, repulsive point p_+ and atttractive circle D_-. Passing to a subsequence, $g_n^{-1}(q)$ converges thus to a point $q_\infty \in D_-$. Furthermore $q_\infty = p$ is impossible because $g_n(g_n^{-1}(q)) = q$ would converge to a point of $\hat{g}_\infty(p)$, which would imply $q \in D_\infty \cap \hat{g}_\infty(p) = \{p_\infty\}$ since $q \in D_\infty$,

contradicting our hypothesis on q. Hence $D_n = [p_n, g_n^{-1}(q)]$ is for n large enough a projective line converging to D_-. Finally $x_n = (p_n, D_n) \in \mathbf{X}$ converges to x with $g_n(x_n)$ converging to x_∞, so $x_\infty \in \mathcal{D}_{(g_n)}(x)$. This shows $\mathcal{S}_{\beta,\alpha}(\hat{g}_\infty(p)) \setminus \mathcal{C}_\beta(\hat{g}_\infty(p)) \subset \mathcal{D}_{(g_n)}(x)$ which concludes the proof of the equality since $\mathcal{D}_{(g_n)}(x)$ is closed.
5. This is a direct consequence of the ρ_∞-equivariance of $\hat{g}_\infty$ proved in Lemma 2.9. □

2.5.3. *Unbalanced type β.* For $g_n \in \mathrm{PGL}_3(\mathbb{R})$ a sequence going simply to infinity of unbalanced type β, with repulsive point p_- and attractive projective line D_+ in $\mathbb{RP}^2$, we define $\mathcal{C}_\alpha^- = \mathcal{C}_\beta(p_-)$, $\mathcal{S}_{\alpha,\beta}^- = \mathcal{S}_{\alpha,\beta}(p_-)$, $\mathcal{C}_\beta^+ = \mathcal{C}_\beta(D_+)$ and $\mathcal{S}_{\beta,\alpha}^+ = \mathcal{S}_{\beta,\alpha}(D_+)$.

Lemma 2.18. *There exists a surjective application $\phi\colon \mathbf{X} \to \mathcal{C}_\beta^+$ such that:*

1. *$\phi|_{\mathbf{X}\setminus\mathcal{C}_\alpha^-}$ is a (smooth) $(\mathbf{S}^1 \times \mathbb{R})$-fiber bundle whose fibers are the $\mathcal{S}_{\beta,\alpha}(x) \setminus \mathcal{C}_\alpha^-$ for $x \in \mathcal{C}_\alpha^-$, but ϕ is not continuous on $\mathcal{C}_\alpha^-$;*
2. *for $x \in \mathbf{X} \setminus \mathcal{S}_{\alpha,\beta}^-$, $\mathcal{D}_{(g_n)}(x) = \phi(x)$;*
3. *for $x \in \mathcal{S}_{\alpha,\beta}^- \setminus \mathcal{C}_\alpha^-$, $\mathcal{D}_{(g_n)}(x) = \mathcal{C}_\alpha(\phi(x))$;*
4. *for $x \in \mathcal{C}_\alpha^-$, $\mathcal{D}_{(g_n)}(x) = \mathcal{S}_{\alpha,\beta}(\phi(x))$;*
5. *ϕ is equivariant for the morphism $\rho_\infty\colon \mathrm{Stab}(p_-) \to \mathrm{Stab}(p_-) \cap \mathrm{Stab}(D_+)$ of Lemma 2.10.*

Proof. The standard decomposition (2.5) and the relation (2.3) allow us to assume that $g_n \in A^+$ to prove these assumptions. Thus $D_+ = [e_1, e_2]$ and $p_- = [e_3]$ according to Lemma 2.9. The dual application κ of $\mathbf{X}$ being equivariant for the morphism $g \mapsto {}^t g^{-1}$ (see (2.13)), we have $g_n = \kappa \circ g_n^{-1} \circ \kappa^{-1}$ and thus

$$\mathcal{D}_{(g_n)}(x) = \kappa(\mathcal{D}_{(g_n^{-1})}(\kappa^{-1}(x))) \tag{2.14}$$

for any $x \in \mathbf{X}$. Now (g_n^{-1}) goes simply to infinity with unbalanced type α, and $p_+(g_n^{-1}) = [e_3]$, $D_-(g_n^{-1}) = [e_1, e_2]$. Denoting by $\psi\colon \mathbf{X} \to \mathcal{C}_\alpha[e_3]$ the application associated to (g_n^{-1}) in Lemma 2.17 we define $\phi = \kappa \circ \psi \circ \kappa^{-1}$, and all the claims are now a direct consequence of the corresponding statements in Lemma 2.17, thanks to relation (2.14). With $\bar{g}_\infty\colon \mathbb{RP}^2 \setminus \{p_-\} \to D_+$ the application introduced in Lemma 2.10 and denoting $x = (p, D)$, a straightforward calculation in the case of $g_n \in A^+$ furthermore shows that:

- $\phi(x) = (\bar{g}_\infty(p), D_+)$ if $x \notin \mathcal{C}_\alpha^-$,
- and $\phi(x) = (\bar{g}_\infty(D \cap p_-^\perp), D_+)$ if $x \in \mathcal{C}_\alpha^-$. □

Remark 2.19. There is a simple geometric interpretation of the fibration ϕ associated to a sequence (g_n) of unbalanced type β. For any $p \in \mathbb{RP}^2$, $\mathbb{RP}^2 \setminus \{p\}$ is foliated by the intervals $D \setminus \{p\}$ for $D \in (p)^*$, and $\mathbf{X} \setminus \mathcal{C}_\alpha(p)$ is thus foliated by the $\mathcal{S}_{\beta,\alpha}(D) \setminus \mathcal{C}_\alpha(p)$ for $D \in (p)^*$. In other words, $\mathbf{X} \setminus \mathcal{C}_\alpha(p)$ is foliated by the $\mathcal{S}_{\beta,\alpha}(x) \setminus \mathcal{C}_\alpha(p)$ for $x = (p, D) \in \mathcal{C}_\alpha(p)$ (these are cylinders of $\mathbf{X}$). These leaves are precisely the fibers of the fibration defined in Lemma 2.18 with p the repulsive point p_- of the sequence (g_n). Moreover for any $D \in \mathbb{RP}^2_*$ that does not contain p, each of these leaves intersects $\mathcal{C}_\beta(D)$ in one point. If the attractive line D_+ of (g_n) does not contain p_- (which is the case if $g_n \in A^+$) then $\phi(x) = \phi^{-1}(x) \cap \mathcal{C}_\beta^+$ for any $x \in \mathbf{X} \setminus \mathcal{C}_\alpha^-$. It is easy to deduce from this the corresponding description for the fibration associated to a sequence of unbalanced type α.

2.5.4. *Balanced type.* For $g_n \in \mathrm{PGL}_3(\mathbb{R})$ a sequence going simply to infinity of balanced type, with repulsive and attractive points and projective lines $p_- \in D_-$ and $p_+ \in D_+$ in $\mathbb{RP}^2$, we define $x^- = (p_-, D_-)$, $x^+ = (p_+, D_+)$, $\mathcal{C}_\alpha^- = \mathcal{C}_\alpha(p_-)$, $\mathcal{C}_\beta^- = \mathcal{C}_\beta(D_-)$, $\mathcal{S}_{\alpha,\beta}^- = \mathcal{S}_{\alpha,\beta}(p_-)$, $\mathcal{S}_{\beta,\alpha}^- = \mathcal{S}_{\beta,\alpha}(D_-)$, $\mathcal{C}_\alpha^+ = \mathcal{C}_\alpha(p_+)$, $\mathcal{C}_\beta^+ = \mathcal{C}_\beta(D_+)$, $\mathcal{S}_{\alpha,\beta}^+ = \mathcal{S}_{\alpha,\beta}(p_+)$, $\mathcal{S}_{\beta,\alpha}^+ = \mathcal{S}_{\beta,\alpha}(D_+)$, $B_{\alpha\beta}^- = \mathcal{C}_\alpha^- \cup \mathcal{C}_\beta^-$ and $B_{\alpha\beta}^+ = \mathcal{C}_\alpha^+ \cup \mathcal{C}_\beta^+$.

Remark 2.20. Note that $\mathcal{S}_{\alpha,\beta}^- \cap \mathcal{S}_{\beta,\alpha}^- = B_{\alpha\beta}^-$ and $\mathcal{S}_{\alpha,\beta}^+ \cap \mathcal{S}_{\beta,\alpha}^+ = B_{\alpha\beta}^+$.

Lemma 2.21. *For $x \in \mathbf{X} \setminus B_{\alpha\beta}^-$, $\mathcal{D}_{(g_n)}(x) \subset B_{\alpha\beta}^+$. More precisely:*

1. *For $x \in \mathbf{X} \setminus (\mathcal{S}_{\beta,\alpha}^- \cup \mathcal{S}_{\alpha,\beta}^-)$, $\mathcal{D}_{(g_n)}(x) = x^+$.*
2. *For $x \in \mathcal{S}_{\alpha,\beta}^- \setminus \mathcal{S}_{\beta,\alpha}^- = \mathcal{S}_{\alpha,\beta}^- \setminus (\mathcal{C}_\alpha^- \cup \mathcal{C}_\beta^-)$, $\mathcal{D}_{(g_n)}(x) = \mathcal{C}_\alpha^+$.*

3. For $x \in \mathcal{S}^-_{\beta,\alpha} \setminus \mathcal{S}^-_{\alpha,\beta} = \mathcal{S}^-_{\beta,\alpha} \setminus (\mathcal{C}^-_\alpha \cup \mathcal{C}^-_\beta)$*,* $\mathcal{D}_{(g_n)}(x) = \mathcal{C}^+_\beta$*.*
4. For $x \in \mathcal{C}^-_\alpha \setminus \{x^-\}$*,* $\mathcal{D}_{(g_n)}(x) = \mathcal{S}^+_{\alpha,\beta}$*.*
5. For $x \in \mathcal{C}^-_\beta \setminus \{x^-\}$*,* $\mathcal{D}_{(g_n)}(x) = \mathcal{S}^+_{\beta,\alpha}$*.*
6. $\mathcal{D}_{(g_n)}(x^-) = \mathbf{X}$*.*

Proof. The direct inclusions of these claims are direct consequences of Lemmas 2.11 and 2.16. We thus only prove the reverse inclusions, denoting $x = (p, D)$.
1. Since $\mathcal{D}_{(g_n)}(x)$ is non-empty, nothing remains to be proved.
2. If $p_- \in D$, Lemma 2.16 gives for any $D_\infty \in (p_+)^*$ a sequence D_n converging to D such that $\lim g_n(D_n) = D_\infty$. For n large enough, $p_n = D_n \cap [p, D^\perp]$ is a sequence of $\mathbb{R}\mathbf{P}^2$ converging to p, hence $x_n = (p_n, D_n) \in \mathbf{X}$ converges to x and verifies $\lim g_n(x_n) = (p_+, D_\infty)$. This shows $(p_+, D_\infty) \in \mathcal{D}_{(g_n)}(x)$ and thus $\mathcal{C}^+_\alpha \subset \mathcal{D}_{(g_n)}(x)$, finishing the proof of the equality.
3. If $p \in D_-$, Lemma 2.11 gives for any $p_\infty \in D_+$ a sequence p_n converging to p such that $\lim g_n(p_n) = p_\infty$. Then with $D_n = [p_n, p^\perp \cap D]$, $x_n = (p_n, D_n)$ converges to x and verifies $\lim g_n(x_n) = (p_\infty, D_+)$. Hence $\mathcal{C}^+_\beta \subset \mathcal{D}_{(g_n)}(x)$, which concludes the proof.
4. We have $x = (p_-, D)$. We choose $x_\infty = (p_\infty, D_\infty) \in \mathcal{S}^+_{\alpha,\beta} \setminus \mathcal{S}^+_{\beta,\alpha} = \mathcal{S}^+_{\alpha,\beta} \setminus (\mathcal{C}^+_\alpha \cup \mathcal{C}^+_\beta)$. According to Lemma 2.16, there exists $D_n \in \mathbb{R}\mathbf{P}^2_*$ converging to D and such that $\lim g_n(D_n) = D_\infty$. Then for n large enough, $q_n = g_n(D_n) \cap [p_\infty, D_\infty^\perp]$ is a sequence of $\mathbb{R}\mathbf{P}^2$ converging to p_∞. Since D_+ is the repulsive line of (g_n^{-1}) according to Remark 2.14, and $p_\infty \notin D_+$, $p_n = g_n^{-1}(q_n)$ converges to the attractive point of (g_n^{-1}), that is p_-. Finally $x_n = (p_n, D_n))$ converges to x and $\lim g_n(x_n) = x_\infty \in \mathcal{D}_{(g_n)}(x)$. This shows that $\mathcal{D}_{(g_n)}(x)$ contains $\mathcal{S}^+_{\alpha,\beta} \setminus \mathcal{S}^+_{\beta,\alpha}$ and thus $\mathcal{S}^+_{\alpha,\beta}$, since $\mathcal{D}_{(g_n)}(x)$ is closed and $\mathcal{C}^+_\alpha \cup \mathcal{C}^+_\beta$ has empty interior.
5. In this case $x = (p, D_-)$. As before we choose $x_\infty = (p_\infty, D_\infty) \in \mathcal{S}^+_{\beta,\alpha} \setminus \mathcal{S}^+_{\alpha,\beta} = \mathcal{S}^+_{\beta,\alpha} \setminus (\mathcal{C}^+_\beta \cup \mathcal{C}^+_\alpha)$. According to Lemma 2.11 there exists a sequence p_n converging to p such that $\lim g_n(p_n) = p_\infty$, and we define $L_n = [g_n(p_n), p_\infty^\perp \cap D_\infty]$, converging to D_∞. According to Lemma 2.16, $(p_+)^*$ is the repulsive dual projective line of (g_n^{-1}) acting on $\mathbb{R}\mathbf{P}^2_*$. Since $p_+ \notin D_\infty$, $D_n := g_n^{-1}(L_n)$ converges thus to the attractive line of (g_n^{-1}), equal to D_- as we saw in Remark 2.14. Hence $x_n = (p_n, D_n)$ converges to x and $\lim g_n(x_n) = x_\infty \in \mathcal{D}_{(g_n)}(x)$. As before, this concludes the proof of the claim since $\mathcal{D}_{(g_n)}(x)$ is closed.
6. Let $D_\infty \notin (p_+)^*$ and $p_\infty \in D_\infty$. According to Lemma 2.11 there exists (p_n) converging to p_- such that $\lim g_n(p_n) = p_\infty$. For n large enough, $L_n = [g_n(p_n), p_\infty^\perp \cap D_\infty]$ is a projective line converging to D_∞. Since $D_\infty \notin (p_+)^*$, $D_n = g_n^{-1}(L_n)$ converges to D_- and $x_n = (p_n, D_n)$ converges vers x_- with $\lim g_n(x_n) = (p_\infty, D_\infty)$. This proves that $X \setminus \mathcal{S}^+_{\alpha,\beta} \subset \mathcal{D}_{(g_n)}(x_-)$, proving our claim since $\mathcal{D}_{(g_n)}(x^-)$ is closed and $\mathcal{S}^+_{\alpha,\beta}$ has empty interior. □

Example 2.22. Let $g \in \mathrm{PGL}_3(\mathbb{R})$ be a loxodromic element and p_-, $p_\pm$, p_+ be its repulsive, saddle and attractive points (see Example 2.12). Then $x^- = (p_-, [p_-, p_\pm])$ and $x^+ = (p_+, [p_\pm, p_+])$ will respectively be called the *repulsive* and *attractive flags* of g, and

$$B^-_{\alpha\beta}(g) = \mathcal{C}_\alpha(p_-) \cup \mathcal{C}_\beta[p_-, p_\pm], B^+_{\alpha\beta}(g) = \mathcal{C}_\alpha(p_+) \cup \mathcal{C}_\beta[p_\pm, p_+]$$

its *repulsive* and *attractive bouquets of circles*. Those are indeed the repulsive and attractive bouquets of circles of any subsequence of (g^n) going simply to infinity.

Remark 2.23. If (g_n) is of unbalanced type α (respectively β), then we saw in Lemma 2.8 that (g_n^{-1}) is of unbalanced type β (resp. α). Actually, dynamics of (g_n^{-1}) and (g_n) are directly related through the following relations between their dynamical objects: $\mathcal{S}^-_{\alpha,\beta}(g_n^{-1}) = \mathcal{S}^+_{\alpha,\beta}(g_n)$, $\mathcal{C}^-_\alpha(g_n^{-1}) = \mathcal{C}^+_\alpha(g_n)$, $\mathcal{S}^+_{\beta,\alpha}(g_n^{-1}) = \mathcal{S}^-_{\beta,\alpha}(g_n)$, $\mathcal{C}^+_\beta(g_n^{-1}) = \mathcal{C}^-_\beta(g_n)$. If (g_n) is of balanced type, then (g_n^{-1}) is also of balanced type according to Lemma 2.8, and in this case any attractive (respectively repulsive) object of (g_n) is the corresponding repulsive (resp. attractive) object of (g_n^{-1}). For instance $\mathcal{C}^-_\alpha(g_n^{-1}) = \mathcal{C}^+_\alpha(g_n)$, $\mathcal{S}^-_{\alpha,\beta}(g_n^{-1}) = \mathcal{S}^+_{\alpha,\beta}(g_n)$.

3. Fundamental domains in the flag space

In this section we introduce a natural notion of Schottky subgroups of $\mathrm{PGL}_3(\mathbb{R})$, for which we describe fundamental domains and limit sets in the flag space.

3.1. **Fundamental domain for a loxodromic element.** Let g be a loxodromic element of $\mathrm{PGL}_3(\mathbb{R})$ having positive eigenvalues, whose attractive (respectively repulsive) bouquet of circles is denoted by $B^+_{\alpha\beta}$ (resp. $B^-_{\alpha\beta}$), and let (g^t) be the one-parameter loxodromic subgroup of $\mathrm{PGL}_3(\mathbb{R})$ for which $g = g^1$. We denote by Γ the subgroup generated by g and we introduce the open set

$$\Omega \coloneqq \mathbf{X} \setminus (B^-_{\alpha\beta} \cup B^+_{\alpha\beta})$$

of $\mathbf{X}$. We will say that an open set $U \subset \Omega$ is a *fundamental domain for the action of* Γ *on* Ω, if

(a) for any $x \neq y \in U$, $y \notin \Gamma \cdot x$;
(b) $\bigcup\limits_{\gamma\in\Gamma} \gamma(\bar{U}) = \Omega$.

Lemma 3.1. *There exists a compact neighbourhood H^- of $B^-_{\alpha\beta}$, as close to $B^-_{\alpha\beta}$ as we want, disjoint from $B^+_{\alpha\beta}$, and satisfying the following properties.*

1. *H^- is a genus two handlebody, whose boundary is transverse to the orbits of (g^t).*
2. *Denoting $H^+ \coloneqq \mathbf{X} \setminus \mathrm{Int}(g(H^-))$, $(g^{-n}(H^-))$ and $(g^n(H^+))$ respectively converge to $B^-_{\alpha\beta}$ and $B^+_{\alpha\beta}$ for the Hausdorff topology.*
3. *$\Phi\colon (x,t) \in \partial H^- \times \mathbb{R} \mapsto g^t(x) \in \Omega$ is a diffeomorphism.*
4. *$U \coloneqq \mathbf{X} \setminus (H^- \cup H^+)$ is a fundamental domain for the action of Γ on Ω.*

We recall that a *genus two handlebody* is a (unique up to homeomorphism) connected compact and orientable three-manifold with boundary, obtained from the three-ball after adding two 1-handles. The boundary of a genus two handlebody is homeomorphic to the closed connected and orientable surface of genus two.

Proof of Lemma 3.1. 1. Up to conjugation in $\mathrm{PGL}_3(\mathbb{R})$, we can assume that

$$g^t = \begin{bmatrix} \mathrm{e}^{\alpha t} & 0 & 0 \\ 0 & \mathrm{e}^{\beta t} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $\alpha > \beta > 0$, so that the repulsive circles of g are $\mathcal{C}^-_\alpha = \mathcal{C}_\alpha[e_3]$ and $\mathcal{C}^-_\beta = \mathcal{C}_\beta[e_2, e_3]$. We introduce $x^- = ([e_3], [e_2, e_3]) = \mathcal{C}^-_\alpha \cap \mathcal{C}^-_\beta$ and the open g^t-invariant set $\mathcal{U} = \mathbf{X} \setminus (\mathcal{S}_{\beta,\alpha}[e_1, e_2] \cup \mathcal{S}_{\alpha,\beta}[e_1])$. In the chart $\psi\colon ([x, y, 1], [(x, y, 1), (z, 1, 0)]) \in \mathcal{U} \mapsto (x, y, z) \in \mathbb{R}^3$ of $\mathcal{U}$, a straightforward calculation shows that (g^t) is conjugated to the diagonal flow

$$a^t \coloneqq \mathrm{Diag}(\mathrm{e}^{\alpha t}, \mathrm{e}^{\beta t}, \mathrm{e}^{(\alpha-\beta)t}) = \psi \circ g^t \circ \psi^{-1}.$$

We first build a neighbourhood of $\mathcal{C}^-_\alpha$ transverse to (g^t). Let D be a closed disk of $\mathbb{R}^2$ centered at the origin whose boundary is transverse to the diagonal flow $\mathrm{Diag}(\mathrm{e}^{\alpha t}, \mathrm{e}^{\beta t})$. Then $A_0 = \{\psi^{-1}(x, y, z) \mid (x, y) \in D, z \in \mathbb{R}\}$ is transverse to (g^t) and is a neighbourhood of $\mathcal{C}^-_\alpha \cap \mathcal{U} = \psi^{-1}(\{0\}^2 \times \mathbb{R})$. The solid torus $A = \{([x : y : 1], [(x, y, 1), q]) \mid (x, y) \in D, q \in [e_1, e_2]\}$ is a neighbourhood of $\mathcal{C}^-_\alpha$, it is the closure of A_0. Since $\partial(A \setminus A_0) = \{([x : y : 1], [(x, y, 1), e_1]) \mid (x, y) \in \partial D\}$ is transverse to the orbits of (g^t) and (g^t) preserves $\mathcal{U}$, A is transverse to (g^t). Choosing at the beginning the disk D as little as we want, A is as close to $\mathcal{C}^-_\alpha$ as we want. Since $\mathcal{C}_\alpha[e_1]$ is the repulsive circle of (g^{-t}), this discussion applied to (g^{-t}) provides us with a solid torus C transverse to (g^{-t}) which is a neighbourhood of $\mathcal{C}_\alpha[e_1]$. Its image by the dual application of $\mathbf{X}$ introduced in (2.13) is thus a solid torus $C' = \kappa(C)$ which is a neighbourhood of $\mathcal{C}_\beta[e_2, e_3]$, transverse to (g^t) by equivariance of κ (see (2.9)). We can moreover choose C' as close to $\mathcal{C}_\beta[e_2, e_3]$ as we want. We now only need to choose a little closed ball B centered at x^- and transverse to (g^t), and to glue to B the handles A and C', to obtain a neighbourhood H^- of $B^-_{\alpha\beta}$ transverse to (g^t). By construction this neighbourhood is homeomorphic to a genus two handlebody, which concludes the proof of the claim.

2. Let $p_-, p_\pm, p_+$ be the attractive, saddle and repulsive points of g in $\mathbb{R}\mathbf{P}^2$. Since $x_1 = (p_-, [p_-, p_+]) \in \mathcal{S}^+_{\alpha,\beta}(g) \setminus \mathcal{S}^+_{\beta,\alpha}(g)$, $\mathcal{D}_{(g^{-n})}(x_1) = \mathcal{C}^-_\alpha$ according to Lemma 2.21. Since $x_2 = (p_\pm, [p_\pm, p_-]) \in \mathcal{S}^+_{\beta,\alpha}(g) \setminus \mathcal{S}^+_{\alpha,\beta}(g)$, $\mathcal{D}_{(g^{-n})}(x_2) = \mathcal{C}^-_\beta$. Since $x_1 \in \mathcal{C}^-_\alpha \subset \mathrm{Int}\, H^-$ and $x_2 \in \mathcal{C}^-_\beta \subset \mathrm{Int}\, H^-$, we thus obtain $B^-_{\alpha\beta} \subset \cup_{x\in \mathrm{Int}\, H^-} \mathcal{D}_{(g^{-n})}(x)$. Applying Lemma 2.21 and Remark 2.23 to

(g^{-n}), we also have $\cup_{x\in H^-}\mathcal{D}_{(g^{-n})}(x)\subset B^-_{\alpha\beta}$ since $H^-\subset \mathbf{X}\backslash B^+_{\alpha\beta}$. Hence $B^-_{\alpha\beta}=\cup_{x\in H^-}\mathcal{D}_{(g^{-n})}(x)=\cup_{x\in \mathrm{Int}\, H^-}\mathcal{D}_{(g^{-n})}(x)$, and according to Lemma 2.2 this implies that $(g^{-n}(H^-))$ converges to $B^-_{\alpha\beta}$. We show on the same way that $(g^n(H^+))$ converges to $B^+_{\alpha\beta}$.
3. Since the orbits of (g^t) are transverse to ∂H^- and escape out from H^-, Φ is a local diffeomorphism. Moreover a g^t-orbit cannot cross ∂H^- more than once, hence Φ is injective. The description of the dynamics of (g^n) in the previous claim shows its surjectivity, finishing the proof of the claim.
4. This is a direct consequence of the previous claim. □

Proposition 3.2. *1. Γ acts freely, properly and cocompactly on Ω.*
2. Furthermore $\Gamma\backslash\Omega$ is diffeomorphic to the product of the circle with the closed connected and orientable surface of genus two.

Proof. 1. No pair of points of Ω being dynamically related according to Lemma 2.21, Lemma 2.1 implies that the action of Γ on Ω is proper. This action is free since any non-trivial element of Γ has all its fixed points on $B^-_{\alpha\beta}\cup B^+_{\alpha\beta}$. Finally, this action is cocompact since we found a relatively compact fundamental domain $U\subset\Omega$ for the action of Γ.
2. According to the third claim of Lemma 3.1, $\Gamma\backslash\Omega$ is indeed diffeomorphic to the quotient of $\partial H^-\times[0\,;1]$ by the equivalence relation $(x,0)\sim(x,1)$, and thus to $\partial H^-\times\mathbf{S}^1$. □

3.2. **Fundamental domains for Schottky subgroups.** We now introduce a notion of Schottky subgroups in $\mathrm{PGL}_3(\mathbb{R})$. Let us first recall that two flags (p,D) and (p',D') in $\mathbf{X}$ are *in general position* if $p\notin D'$ and $p'\notin D$.

Definition 3.3. We will say that $d\geq 1$ loxodromic elements $g_1,\dots,g_d$ of $\mathrm{PGL}_3(\mathbb{R})$ are *in general position* if their attractive and repulsive flags $\{x_i^\pm\}$ are pairwise in general position.

Note that the bouquets of circles of loxodromic elements in general position are pairwise disjoint.

Proposition-Definition 3.4. Let $g_1,\dots,g_d$ be loxodromic elements of $\mathrm{PGL}_3(\mathbb{R})$ in general position, whose repulsive and attractive bouquet of circles are denoted by $B_1^\pm,\dots,B_d^\pm$. Then up to replacing each g_i by $g_i^{r_i}$ for $r_i>0$ large enough, $g_1,\dots,g_d$ satisfy the following: there exists $2d$ pairwise disjoint compact genus two handelbodies $\{H_1^-,H_1^+,\dots,H_d^-,H_d^+\}$ in $\mathbf{X}$, such that each $H_i^\pm$ is a neighbourhood of $B_i^\pm$ and $H_i^+=\mathbf{X}\setminus\mathrm{Int}\,g_i(H_i^-)$. We will say in this case that $\Gamma=\langle g_1,\dots,g_d\rangle$ is a *Schottky subgroup* of $\mathrm{PGL}_3(\mathbb{R})$, and that $\{H_i^\pm\}_{i=1}^d$ is a *set of separating handelbodies* for the g_i.

Proof. Since the statement is claimed modulo finite iterates of the g_i, we can assume that each of them has positive eigenvalues. For any i, the compact genus two handlebody neighbourhood H_i^- of B_i^- built in Lemma 3.1 can be chosen as close to B_i^- as we want, possibly replacing g_i by an iterate $g_i^{r_i}$. According to Lemma 3.1, $H_i^+\coloneqq\mathbf{X}\setminus\mathrm{Int}(g_i(H_i^-))$ is a compact genus two handlebody, neighbourhood of the attractive bouquet of circles B_i^+ of g_i, such that $g^n(H_i^+)$ converges to B_i^+. We can thus choose H_i^+ as close to B_i^+ as we want, possibly replacing again g_i by an iterate. Since the $B_i^\pm$ are disjoint, the $H_i^\pm$ are also pairwise disjoint if they are sufficiently close to the $B_i^\pm$. □

Remark 3.5. Note that Lemma 2.21 shows that the repulsive and attractive bouquet of circles of a loxodromic element g are the only geometric objects with respect to which (g^n) has a North-South dynamics with repulsive and attractive sets *of equal dimensions*. The notion of Schottky subgroups defined previously is in this sense imposed by the dynamics in $\mathbf{X}$ of loxodromic elements of $\mathrm{PGL}_3(\mathbb{R})$. Indeed, Definition 3.4 is the natural translation of the classical definition of a Schottky subgroup Γ_0 of $\mathrm{PSL}_2(\mathbb{R})$, where the half-planes of $\overline{\mathbf{H}^2}$ containing the repulsive and attractive points in $\partial\mathbf{H}^2$ of the loxodromic generators of Γ_0 are here replaced by the handlebodies containing the repulsive and attractive bouquet of circles of the generators of Γ.

By construction, these Schottky subgroups satisfy the classical "ping-pong" Lemma.

Proposition 3.6. *Let $\Gamma = \langle g_1, \dots, g_d\rangle$ be a Schottky subgroup of* $\mathrm{PGL}_3(\mathbb{R})$ *and $H_i^{\pm}$ be a set of separating handlebodies for the g_i.*

1. *$\Gamma = \langle g_1, \dots, g_d\rangle$ is a discrete subgroup of $\mathrm{PGL}_3(\mathbb{R})$ freely generated by $g_1, \dots, g_d$.*
2. *With $U = \bigcap_{i=1}^{d}(\mathbf{X} \setminus (H_i^- \cup H_i^+))$, $\Omega = \bigcup_{\gamma\in\Gamma} \gamma(\bar{U})$ is an open set of $\mathbf{X}$ where Γ acts freely, properly and cocompactly.*

Proof. According to Proposition 3.2, the subgroups $\langle g_i\rangle$ acts freely, properly and cocompactly on $\mathbf{X} \setminus (B_i^- \cup B_i^+)$, with $U_i = \mathbf{X} \setminus (H_i^- \cup H_i^+)$ as a fundamental domain. These properties allow us to apply the classical ping-pong Lemma to the action of Γ on Ω. More precisely, the version of *Klein's combination Theorem* proved by Frances in [Fra04, Theorem 5] (see also [Mas88]) allows us to conclude: Ω is open and Γ is a discrete free subgroup acting freely, properly and cocompactly on Ω. □

3.3. **Limit sets of Schottky subgroups.** In this paragraph, we would like to give the following intrisinc dynamical meaning to the open set Ω found in Proposition 3.6: $\Lambda(\Omega) = \partial\Omega$ is the limit set in $\mathbf{X}$ of the Schottky subgroup $\Gamma = \langle g_1, \dots, g_d\rangle$ of $\mathrm{PGL}_3(\mathbb{R})$.

To this end, we will use the notion of *Gromov boundary* of the free group Γ, denoted by $\partial_\infty\Gamma$. Our use of $\partial_\infty\Gamma$ being very elementary, we define this set and the topology of $\Gamma \cup \partial_\infty\Gamma$ in a simple and naive way to avoid a technicality which would be useless here, and we refer to [GdlH90, Chapitre 6 §1] or [BH99, Chapter III.H §3] for more details (and for the relation, in the case of trees, of the definition that we use here with the definition in terms of half-geodesics). We define $\partial_\infty\Gamma$ as the set of non-empty right-infinite words $g_{i_1}^{\varepsilon_1} \dots g_{i_n}^{\varepsilon_n} \dots$ on the alphabet $\mathcal{A} = \{g_1, g_1^{-1}, \dots, g_d, g_d^{-d}\}$ (with $i_k \in \{1, \dots, d\}$ and $\varepsilon_i \in \{\pm 1\}$) that are reduced, that is $\varepsilon_k i_k \neq -\varepsilon_{k+1} i_{k+1}$. We will say that a reduced word $g_{i_1}^{\varepsilon_1} \dots g_{i_n}^{\varepsilon_n} \in \Gamma$ has *length* $n = |\gamma|$. The length of words $|\gamma|$ defines the *word metric* $d(\gamma, \delta) = |\gamma\delta^{-1}|$ on Γ. The natural embedding of $\partial_\infty\Gamma$ in $\mathcal{A}^{\mathbb{N}}$ defined by identifying $g_{i_1}^{\varepsilon_1} \dots g_{i_n}^{\varepsilon_n} \dots$ to the sequence $(g_{i_n}^{\varepsilon_n})$ endows $\partial_\infty\Gamma$ with the restriction of the product topology of $\mathcal{A}^{\mathbb{N}}$, for which $\partial_\infty\Gamma$ is a Cantor space (in particular, $\partial_\infty\Gamma$ is compact). The disjoint union $\Gamma \cup \partial_\infty\Gamma$ is endowed with a (metrizable) topology extending those of Γ and $\partial_\infty\Gamma$ and for which $\Gamma \cup \partial_\infty\Gamma$ is a compact space. For this topology, a sequence $\gamma_k \in \Gamma$ such that $|\gamma_k| \to +\infty$ converges to a point $\delta_\infty = g_1^{\varepsilon_1} \dots g_n^{\varepsilon_n} \dots \in \partial_\infty\Gamma$ if, and only if there exists a non-decreasing sequence of non-negative integers $n_k \to +\infty$ and a sequence $\mu_k \in \Gamma$ such that $\gamma_k = \delta_{n_k}\mu_k$, where $\delta_n = g_1^{\varepsilon_1} \dots g_n^{\varepsilon_n}$ denotes the sequence of finite subwords of δ_∞.

Let $\{H_i^{\pm}\}_{i=1}^d$ be a set of separating handlebodies for the g_i as defined in Definition 3.4. We associate to any $\gamma = g_{i_1}^{\varepsilon_1} \dots g_{i_n}^{\varepsilon_n} \in \Gamma$ the compact $K(\gamma) \coloneqq g_{i_1}^{\varepsilon_1} \dots g_{i_{n-1}}^{\varepsilon_{n-1}}(H_{i_n}^{\varepsilon_n})$ and to any $\gamma_\infty \in \partial_\infty\Gamma$ the sequence of compacts $K(\gamma_n)$ with γ_n the finite subwords of γ_∞. Since $g_i^{\varepsilon}(H_j^{\delta}) \subset \mathrm{Int}(H_i^{\varepsilon})$ for any $(j, \delta) \neq (i, -\varepsilon)$, we have $g_{i_n}^{\varepsilon_n}(H_{i_{n+1}}^{\varepsilon_{n+1}}) \subset H_{i_n}^{\varepsilon_n}$ for any n, and the sequence of compacts $K(\gamma_n)$ associated to γ_∞ is thus decreasing. Therefore, the intersection

$$B_{\alpha\beta}^+(\gamma_\infty) \coloneqq \bigcap_{n\in\mathbb{N}} K(\gamma_n) \tag{3.1}$$

is a non-empty connected compact subset, which is the limit of $(K(\gamma_n))$ for the Hausdorff topology.

The following statement and some parts of its proof were inspired by analog results proved in [Fra04, Lemma 7 and 8] for Schottky conformal subgroups acting on the Einstein universe. We also refer to [KLP17, §6] for related results in the general setting of regular discrete subgroups of semi-simple Lie groups.

Proposition 3.7. *Let $\Gamma = \langle g_1, \dots, g_d\rangle$ be a Schottky subgroup of $\mathrm{PGL}_3(\mathbb{R})$, $\{H_i^{\pm}\}_{i=1}^d$ be a set of separating handlebodies for the g_i and Ω be the open set of discontinuity of Proposition 3.6 defined by the $H_i^{\pm}$.*

1. *The application $B_{\alpha\beta}^+$ defined in (3.1) is an homeomorphism from $\partial_\infty\Gamma$ to the set of connected components of $\Lambda(\Gamma) \coloneqq \partial\Omega$ endowed with the Hausdorff topology.*
2. *Let $\gamma_\infty \in \partial_\infty\Gamma$ and (γ_n) denote its sequence of finite subwords. Then any subsequence of (γ_n) going simply to infinity is of balanced type with attractive bouquet of circles $B_{\alpha\beta}^+(\gamma_\infty)$.*

3. *Let* (γ_n) *be a sequence of* Γ *going simply to infinity in* $\mathrm{PGL}_3(\mathbb{R})$*. Then* (γ_n) *converges in* $\Gamma \cup \partial_\infty\Gamma$ *to a point* $\gamma_\infty \in \partial_\infty\Gamma$*,* (γ_n) *is of balanced type and* $B^+_{\alpha\beta}(\gamma_n) = B^+_{\alpha\beta}(\gamma_\infty)$*.*
4. Ω *is equal to the* maximal open subset of discontinuity *of* Γ *defined as:*

$$\Omega(\Gamma) \coloneqq \mathbf{X} \setminus \bigcup_{\substack{\gamma_n \in \Gamma \\ \gamma_n \underset{simply}{\longrightarrow} \infty}} B^+_{\alpha\beta}(\gamma_n).$$

Proof. 1. The proof of this first claim is formally the same than [Fra04, Lemma 7] but we repeat it here for the convenience of the reader. We define $U = \mathbf{X} \setminus \bigcup_i (H_i^- \cup H_i^+)$ as in Proposition 3.6, and we introduce

$$\Omega_n = \cup_{|\gamma| \leq n} \gamma(\bar{U}) \text{ and } \Lambda_n = \mathbf{X} \setminus \Omega_n,$$

so that $\Lambda(\Gamma) = \cap_n \Lambda_n$. Note that for any $\gamma_\infty \in \partial_\infty\Gamma$, $K(\gamma_n) \subset \Lambda_n$ for any n, with γ_n the finite subwords of γ_∞, showing that $B^+_{\alpha\beta}(\gamma_\infty) \subset \Lambda(\Gamma)$. Let $x \in \Lambda(\Gamma)$, and C_n, C respectively denote the connected components of x in Λ_n and $\Lambda(\Gamma)$. Any connected component of Λ_n is of the form $K(\gamma)$ for some $\gamma \in \Gamma$ of length n. Moreover C_n is decreasing, hence $C_n = K(\gamma_n)$ with γ_n the finite subwords of some $\gamma_\infty \in \partial_\infty\Gamma$. Since $C \subset C_n$ for any n, we have $C \subset B^+_{\alpha\beta}(\gamma_\infty) = \cap_n C_n$ and by maximality of the connected component this inclusion is an equality. This shows that $B^+_{\alpha\beta}(\gamma_\infty)$ is always a connected component of $\Lambda(\Gamma)$ and that $B^+_{\alpha\beta}$ is surjective onto the set of connected components of $\Lambda(\Gamma)$. To prove the injectivity, take $\gamma_\infty \neq \gamma'_\infty$ in $\partial_\infty\Gamma$ of finite subwords (γ_n) and (γ'_n). There exists k such that $g_{i_k}^{\varepsilon_k} \neq g_{i'_k}^{\varepsilon'_k}$ and for any $n \geq k$ $K(g_{i_k}^{\varepsilon_k} \dots g_{i_n}^{\varepsilon_n}) \subset H_{i_k}^{\varepsilon_k}$ and $K(g_{i'_k}^{\varepsilon'_k} \dots g_{i_n}^{\varepsilon_n}) \subset H_{i'_k}^{\varepsilon'_k}$ are thus disjoints. This implies that $K(\gamma_n)$ and $K(\gamma'_n)$ are disjoints for n large enough and thus that $B^+_{\alpha\beta}(\gamma_\infty) \neq B^+_{\alpha\beta}(\gamma'_\infty)$.

It only remains to show that $B^+_{\alpha\beta}$ is continuous and the conclusion will follow by compacity of $\partial_\infty\Gamma$. Let $\gamma_\infty^{(n)} \in \partial_\infty\Gamma$ converge to γ_∞ with $(\gamma_k^{(n)})$ and (γ_k) the respective sequences of finite subwords of $\gamma_\infty^{(n)}$ and γ_∞. There exists then a strictly increasing sequence $k_n \to +\infty$ such that $\gamma_{k_n}^{(n)} = \gamma_{k_n}$ for any n, and thus $K(\gamma_{k_n}^{(n)}) = K(\gamma_{k_n})$ with $K(\gamma_{k_n})$ converging to $B^+_{\alpha\beta}(\gamma_\infty)$. Since $K(\gamma_\infty^{(n)}) \subset K(\gamma_{k_n}^{(n)})$, $B^+_{\alpha\beta}(\gamma_\infty)$ is thus the only accumulation point of $(K(\gamma_\infty^{(n)}))$. By compacity of the space of compacts for the Hausdorff topology, this shows that $K(\gamma_\infty^{(n)})$ converges to $B^+_{\alpha\beta}(\gamma_\infty)$ and concludes the proof of the claim.
2. The proof of this claim was inspired by the one of [Fra04, Lemma 8]. We will make a repeated use of the following argument to prove our second claim.

Fact 3.8. *For* $1 \leq i \leq g$*, let* ${H'}_i^-$ *be a compact neighbourhood of* B_i^- *close enough to* H_i^- *for the Hausdorff topology, and let define* ${H'}_i^+ \coloneqq \mathbf{X} \setminus g_i(\mathrm{Int}({H'}_i^-))$*. Then* $U'_i = \mathbf{X} \setminus ({H'}_i^- \cup {H'}_i^+)$ *remains a fundamental domain for the action of* $\langle g_i \rangle$ *on* $\mathbf{X} \setminus (B_i^- \cup B_i^+)$*, and* $U' \coloneqq U'_i \cap_{j \neq i} U_j$ *remains a fundamental domain for the action of* Γ *on* Ω*:* $\cup_{\gamma\in\Gamma}\gamma(\overline{U'}) = \cup_{\gamma\in\Gamma}\gamma(\overline{U}) = \Omega$*.*

Proof. The same proof as in Lemma 3.1 shows that U'_i is a fundamental domain for the action of $\langle g_i \rangle$ on $\mathbf{X} \setminus (B_i^- \cup B_i^+)$. If ${H'}_i^-$ is close enough to H_i^-, then ${H'}_i^-$ et ${H'}_i^+$ remain disjoints from the other $H_j^\pm$ and the proof of Proposition 3.6 applies thus to the neighbourhoods $\{{H'}_i^\pm; H_j^\pm, j \neq i\}$, showing that $\Omega' \coloneqq \cup_{\gamma\in\Gamma}\gamma(\overline{U'})$ is open. Moreover, if ${H'}_i^-$ is close enough to H_i^- then $\overline{U'}$ is close enough to $\overline{U}$ to be contained in its neighbourhood Ω, and then $\Omega' \subset \Omega$ since Ω is Γ-invariant. Likewise, $\overline{U}$ is in this case contained in the neighbourhood Ω' of $\overline{U'}$, showing that $\Omega \subset \Omega'$ and finishing the proof. □

In other words, slight modifications of the neighbourhoods $H_i^\pm$ are authorized. Let (γ_n) be a subsequence of the finite subwords of a point $\gamma_\infty \in \partial_\infty\Gamma$, going simply to infinity in $\mathrm{PGL}_3(\mathbb{R})$. Passing to a subsequence of (γ_n), we can assume that there exists two letters $g_a^{\varepsilon_a}$ and $g_b^{\varepsilon_b} \neq g_a^{-\varepsilon_a}$ such that the reduced word γ_n ends by $g_a^{\varepsilon_a}$ for any n and that $\gamma_n g_b^{\varepsilon_b}$ is a subsequence of the finite subwords of γ_∞. Therefore, $\gamma_n(H_b^{\varepsilon_b}) = K(\gamma_n g_b^{\varepsilon_b})$ converges to $B^+_{\alpha\beta}(\gamma_\infty)$.

Fact 3.9. (γ_n) *is of balanced type.*

Proof. We assume by contradiction that this is not the case. Possibly replacing (γ_n) by (γ_n^{-1}), we can thus assume that (γ_n) is of unbalanced type α according to Lemma 2.8. We denote by $\mathcal{C}^- \subset \mathcal{S}^-$ and $\mathcal{C}^+ \subset \mathcal{S}^+$ its repulsive and attractive circles and surfaces in $\mathbf{X}$ and by $\phi\colon \mathbf{X} \to \mathcal{C}^+$ the fibration introduced in Lemma 2.17.

We first assume by contradiction that $\mathcal{C}^- \subset \mathrm{Int}(H_b^{\varepsilon_b})$. According to Lemma 2.2, $B^+_{\alpha\beta}(\gamma_\infty) = \lim \gamma_n(H_b^{\varepsilon_b})$ would then contain $\bigcup_{x\in\mathcal{C}^-} \mathcal{D}_{(\gamma_n)}(x)$ which is equal to $\bigcup_{y\in\mathcal{C}^+} \mathcal{S}_{\beta,\alpha}(y)$ according to Lemma 2.17 and thus to $\mathbf{X}$. This is impossible since $B^+_{\alpha\beta}(\gamma_\infty) \subset \Lambda(\Gamma)$ is a strict subset of $\mathbf{X}$. Therefore $\mathcal{C}^- \cap \mathrm{Int}(H_b^{\varepsilon_b})$ is a strict subset of $\mathcal{C}^-$ that we assume to be non-empty by contradiction. We can then slightly modify $H_b^{\varepsilon_b}$ to a neighbourhood $H'^{\varepsilon_b}_b$ of $B_b^{\varepsilon_b}$, in such a way that $\mathcal{C}^- \cap H_b^{\varepsilon_b} \subsetneq \mathcal{C}^- \cap \mathrm{Int}(H'^{\varepsilon_b}_b)$, and the new open set U' defined by $H'^{\varepsilon_b}_b$ as in Fact 3.8 remains a fundamental domain of Ω. The first claim of the proposition applies then to $H'^{\varepsilon_b}_b$ and $(\gamma_n(H'^{\varepsilon_b}_b))$ converges thus to a connected component K' of $\Lambda(\Gamma)$. According to Lemmas 2.2 and 2.17, $K' \supset \bigcup_{x\in\mathcal{C}^-\cap\mathrm{Int}(H'^{\varepsilon_b}_b)} \mathcal{S}_{\beta,\alpha}(\phi(x))$ and $B^+_{\alpha\beta}(\gamma_\infty) \subset \mathcal{S}^+ \cup \bigcup_{x\in\mathcal{C}^-\cap H_b^{\varepsilon_b}} \mathcal{S}_{\beta,\alpha}(\phi(x))$. By hypothesis on $H'^{\varepsilon_b}_b$, K' and $B^+_{\alpha\beta}(\gamma_\infty)$ are thus distincts and therefore disjoints as they both are connected components of $\Lambda(\Gamma)$. This contradicts the fact that they both contain $\bigcup_{x\in B_b^{\varepsilon_b}} \mathcal{D}_{(\gamma_n)}(x)$.

Finally $\mathcal{C}^- \cap \mathrm{Int}(H_b^{\varepsilon_b}) = \varnothing$, and possibly shrinking $H_b^{\varepsilon_b}$ thanks to Fact 3.8 we can assume that $\mathcal{C}^- \cap H_b^{\varepsilon_b} = \varnothing$. Since any α-β surface intersects any α-circle, there exists $y \in (\mathcal{S}^+ \setminus \mathcal{C}^+) \cap \mathrm{Int}(H_{i_1}^{\varepsilon_1})$. According to Remark 2.23, $\mathcal{C}^+$, $\mathcal{S}^+$, $\mathcal{C}^-$ and $\mathcal{S}^-$ are respectively the repulsive and attractive circles and surfaces of the sequence (γ_n^{-1}) of unbalanced type β, and we thus have $\mathcal{D}_{(\gamma_n^{-1})}(y) = \mathcal{C}_\alpha(y')$ with $y' \in \mathcal{C}^-$ according to Lemma 2.18. Moreover $\mathcal{C}_\alpha(y') = \mathcal{D}_{(\gamma_n^{-1})}(y) \subset H_a^{\varepsilon_a}$ because $\gamma_n^{-1} = g_a^{-\varepsilon_a} \dots g_{i_1}^{-\varepsilon_1}$ and $y \in \mathrm{Int}(H_{i_1}^{\varepsilon_1})$. Now $\mathcal{S}^- \cap H_b^{\varepsilon_b} \neq \varnothing$ since any β-α surface meets any β-circle but $\mathcal{S}^- \cap H_b^{\varepsilon_b}$ is disjoint from $\mathcal{C}_\alpha(y')$, because $\mathcal{C}_\alpha(y') \subset H_a^{\varepsilon_a}$ and $H_a^{\varepsilon_a}$ is disjoint from $H_b^{\varepsilon_b}$. In other words, denoting $p = \pi_\alpha(y') \in \mathbb{R}\mathbf{P}^2$, the compact set $\pi_\beta(\mathcal{S}^- \cap H_b^{\varepsilon_b})$ of $\mathbb{R}\mathbf{P}^2_*$ is disjoint from $p^* = \{D \in \mathbb{R}\mathbf{P}^2_* \mid D \not\ni p\}$, where π_α and π_β denote the two coordinate projections of $\mathbf{X}$ on $\mathbb{R}\mathbf{P}^2$ and $\mathbb{R}\mathbf{P}^2_*$. As before, this allows us to slightly modify $H_b^{\varepsilon_b}$ into a neighbourhood $H'^{\varepsilon_b}_b$ of $B_b^{\varepsilon_b}$ satisfying the assumptions of Fact 3.8 and such that $\pi_\beta(\mathcal{S}^- \cap H_b^{\varepsilon_b}) \subsetneq \pi_\beta(\mathcal{S}^- \cap \mathrm{Int}(H'^{\varepsilon_b}_b))$. The sequence $(\gamma_n(H'^{\varepsilon_b}_b))$ converges thus to a connected component K' of $\Lambda(\Gamma)$ containing $\bigcup_{x\in\mathcal{S}^-\cap\mathrm{Int}(H'^{\varepsilon_b}_b)} \mathcal{C}_\beta(\phi(x))$ whereas $B^+_{\alpha\beta}(\gamma_\infty) \subset \mathcal{C}^+ \cup \bigcup_{x\in\mathcal{S}^-\cap H_b^{\varepsilon_b}} \mathcal{C}_\beta(\phi(x))$. This shows that $K' \neq B^+_{\alpha\beta}(\gamma_\infty)$ since $\phi(x)$ does only depend on $\pi_\beta(x)$. As before, K' and $B^+_{\alpha\beta}(\gamma_\infty)$ should then be disjoints which contradicts the fact that they both contain $\bigcup_{x\in B_b^{\varepsilon_b}} \mathcal{D}_{(\gamma_n)}(x)$. This final contradiction concludes the proof that (γ_n) is of balanced type. □

We use the notations of Lemma 2.21 for the dynamical objects of the sequence (γ_n) of balanced type. In particular, $B^-_{\alpha\beta}$ and $B^+_{\alpha\beta}$ denote its attractive and repulsive bouquet of circles. Any α-β surface meeting any α-circle of $\mathbf{X}$ there exists a point $y \in (\mathcal{S}^+_{\alpha,\beta} \setminus \mathcal{S}^+_{\beta,\alpha}) \cap \mathrm{Int}(H_{i_1}^{-\varepsilon_1})$, and since $\gamma_n^{-1} = g_a^{-\varepsilon_a} \dots g_1^{-\varepsilon_1}$, $\mathcal{D}_{(\gamma_n^{-1})}(y) \subset H_a^{-\varepsilon_a}$. Since $\mathcal{S}^+_{\alpha,\beta}$ and $\mathcal{S}^+_{\beta,\alpha}$ (respectively $\mathcal{C}^-_\alpha$) are the repulsive surfaces (resp. attractive α-circle) of (γ_n^{-1}), $\mathcal{D}_{(\gamma_n^{-1})}(x) = \mathcal{C}^-_\alpha$ and thus $\mathcal{C}^-_\alpha \subset H_a^{-\varepsilon_a}$. Analog arguments show that $\mathcal{C}^-_\beta \subset H_a^{-\varepsilon_a}$ and $B^-_{\alpha\beta}$ is thus disjoint from $H_b^{\varepsilon_b}$. Therefore $\mathcal{D}_{(\gamma_n)}(x) \subset B^+_{\alpha\beta}$ for any $x \in H_b^{\varepsilon_b}$ and thus $\lim \gamma_n(H_b^{\varepsilon_b}) = B^+_{\alpha\beta}(\gamma_\infty) \subset B^+_{\alpha\beta}$ according to Lemma 2.2. Conversely there exists $y \in (\mathcal{S}^-_{\alpha,\beta} \setminus \mathcal{S}^-_{\beta,\alpha}) \cap \mathrm{Int}(H_b^{\varepsilon_b})$ and thus $\mathcal{D}_{(\gamma_n)}(y) = \mathcal{C}^+_\alpha \subset B^+_{\alpha\beta}(\gamma_\infty)$, and there exists $z \in (\mathcal{S}^-_{\beta,\alpha} \setminus \mathcal{S}^-_{\alpha,\beta}) \cap \mathrm{Int}(H_b^{\varepsilon_b})$ and thus $\mathcal{D}_{(\gamma_n)}(z) = \mathcal{C}^+_\beta \subset B^+_{\alpha\beta}(\gamma_\infty)$. This concludes the proof that $B^+_{\alpha\beta}(\gamma_\infty)$ is the attractive bouquet of circles of any subsequence going simply to infinity of the finite subwords of γ_∞.

3. We now consider a sequence $\gamma_k \in \Gamma$ going simply to infinity in $\mathrm{PGL}_3(\mathbb{R})$. In particular (γ_k) goes to infinity in Γ for the word metric and we consider an accumulation point $\delta_\infty \in \partial_\infty\Gamma$ of (γ_k). Passing to a subsequence we can assume that $\gamma_k = \delta_{n_k}\mu_k$ with (δ_{n_k}) a subsequence of the finite subwords of δ_∞ and $\mu_k \in \Gamma$ always finishing with the same letter $g_a^{\varepsilon_a}$. We choose a letter $g_b^{\varepsilon_b} \neq g_a^{-\varepsilon_a}$ and passing to a subsequence again we can moreover assume that $\gamma_k(H_b^{\varepsilon_b})$ converges for the Hausdorff topology to a compact subset K_∞ of $\mathbf{X}$. We proved previously that $B^+_{\alpha\beta}(\delta_\infty) = \lim K(\delta_{n_k})$ is a bouquet of two circles. Since $\gamma_k(H_b^{\varepsilon_b}) \subset K(\delta_{n_k})$ for any k (because

$g_b^{\varepsilon_b} \neq g_a^{-\varepsilon_a}$) we have $K_\infty \subset B^+_{\alpha\beta}(\delta_\infty)$. Let us assume by contradiction that (γ_k) is of unbalanced type α and denote by $\mathcal{C}^- \subset \mathcal{S}^-$ and $\mathcal{C}^+ \subset \mathcal{S}^+$ its repulsive and attractive circles and surfaces in $\mathbf{X}$ and by $\phi \colon \mathbf{X} \to \mathcal{C}^+$ the fibration introduced in Lemma 2.17. According to Lemma 2.2, K_∞ contains $\bigcup_{x \in (\mathcal{S}^- \setminus \mathcal{C}^-) \cap \mathrm{Int}(H_b^{\varepsilon_b})} \mathcal{C}_\beta(\phi(x))$, and $(\mathcal{S}^- \setminus \mathcal{C}^-) \cap \mathrm{Int}(H_b^{\varepsilon_b})$ being a non-empty open subset of $\mathcal{S}^- \setminus \mathcal{C}^-$ (since any β-α surface intersects any β-circle), its image by ϕ is a non-empty open subset I of $\mathcal{C}^+$ and $\cup_{y \in I} \mathcal{C}_\beta(y)$ contains thus an embedded topological disc D. But then $D \subset K_\infty$ which contradicts $K_\infty \subset B^+_{\alpha\beta}(\gamma_\infty)$. We obtain a contradiction in the same way if we assume (γ_k) to be of unbalanced type β.

Hence (γ_k) is of balanced type and we denote by $\mathcal{S}^-_{\alpha,\beta}$, $\mathcal{S}^-_{\beta,\alpha}$ $\mathcal{C}^+_\alpha$ and $\mathcal{C}^+_\beta$ its repulsive surfaces and attractive circles. As before K_∞ contains $\mathcal{C}^+_\alpha$ since $(\mathcal{S}^-_{\alpha,\beta} \setminus \mathcal{S}^-_{\beta,\alpha}) \cap \mathrm{Int}(H_a^{\varepsilon_a}) \neq \varnothing$ and contains $\mathcal{C}^+_\beta$ since $(\mathcal{S}^-_{\beta,\alpha} \setminus \mathcal{S}^-_{\alpha,\beta}) \cap \mathrm{Int}(H_a^{\varepsilon_a}) \neq \varnothing$. As $K_\infty \subset B^+_{\alpha\beta}(\delta_\infty)$ this proves that $B^+_{\alpha\beta}(\gamma_k) = \mathcal{C}^+_\alpha \cup \mathcal{C}^+_\beta = K_\infty = B^+_{\alpha\beta}(\delta_\infty)$. By injectivity of $x \in \partial_\infty \Gamma \mapsto B^+_{\alpha\beta}(x)$, this shows in particular that δ_∞ is the only accumulation point of (γ_k), which converges thus to δ_∞ by compacity of $\Gamma \cup \partial_\infty \Gamma$. This concludes the proof of our claim.

4. This follows readily from the previous claims. □

We can now summarize our results as follows.

Corollary 3.10. *Let $\Gamma = \langle g_1, \dots, g_d \rangle$ be a Schottky subgroup of $\mathrm{PGL}_3(\mathbb{R})$.*

1. *With $\{H_i^\pm\}_{i=1}^d$ any set of separating handlebodies for the g_i, $\mathbf{X} \setminus \bigcup_{i=1}^d (H_i^- \cup H_i^+)$ is a fundamental domain for the action of Γ on its maximal open subset of discontinuity $\Omega(\Gamma)$.*
2. *Moreover, $\Gamma \backslash \Omega(\Gamma)$ is homeomorphic to a closed three-manifold obtained from the flag space $\mathbf{X}$ after succesively performing d times the following two operations:*
   - *(A) Remove the interior of two disjoint embedded genus two handlebodies H^- and H^+.*
   - *(B) Glue the two boundary components ∂H^- and ∂H^+ of the resulting three-manifold with boundary by a diffeomorphism $f \colon \partial H^- \to \partial H^+$.*

Proof. 1. This is a straightforward reformulation of Propositions 3.6 and 3.7.

2. Let us assume that $d = 2$, that is $\Gamma = \langle g_1, g_2 \rangle$. Then $\Gamma \backslash \Omega$ is homeomorphic to $\sim \backslash \bar{U}$, with $U = \mathbf{X} \setminus \cup_{i=1,2} (H_i^- \cup H_i^+)$ and $\sim$ the equivalence relation generated by the relations $x \sim g_i(x)$ for $i = 1, 2$ and any $x \in \partial H_i^-$. Denoting $\Gamma_1 = \langle g_1 \rangle$ and $\Omega_1 = \mathbf{X} \setminus (B_1^- \cup B_1^+)$, the topology of the quotient $M_1 = \Gamma_1 \backslash \Omega_1$ is obtained from the one of $\mathbf{X}$ by performing the operations (A) and (B) described in the statement. Indeed H_1^- and H_1^+ are disjoint embedded genus two handlebodies in $\mathbf{X}$ and $M_1 = \sim_1 \backslash (\mathbf{X} \setminus (\mathrm{Int}\, H_1^- \cup \mathrm{Int}\, H_1^+))$, where $x \sim_1 g_1(x)$ for any $x \in \partial H_1^-$. If $\pi_1 \colon \Omega_1 \to M_1$ denotes the canonical projection, $H^- = \pi_1(H_2^-)$ and $H^+ = \pi_1(H_2^+)$ are two disjoint embedded genus two handlebodies in M_1. Moreover for any $\bar{x} \in \partial H^-$ there exists an unique $x \in \partial H_2^-$ such that $\bar{x} = \pi_1(x)$, since $\partial H_2^- \subset U_1$ and $\pi_1|_{U_1}$ is injective. This allows to defines a diffeomorphism $f \colon \partial H^- \to \partial H^+$ by $f(\bar{x}) = \pi_1 \circ g_2(x)$, for any $\bar{x} \in \partial H^-$ and $x \in \partial H_2^-$ such that $\bar{x} = \pi_1(x)$. Now $\Gamma \backslash \Omega = \sim \backslash \bar{U}$ is homeomorphic to $\sim_2 \backslash (M_1 \setminus (\mathrm{Int}\, H^- \cup \mathrm{Int}\, H^+))$, where $x \sim_2 f(x)$ for any $x \in \partial H^-$, hence the topology of $\Gamma \backslash \Omega$ is obtained from the one of $\mathbf{X}$ after succesively performing two times the operations (A) and (B). The claim immediately follows by a finite recurrence argument. □

4. Path structures compactifications of geodesic flows

Let $\overline{h}_1, \dots, \overline{h}_d$ be hyperbolic elements of $\mathrm{PSL}_2(\mathbb{R})$ having pairwise distincts repulsive and attractive fixed points in the boundary $\partial \mathbf{H}^2$ of the hyperbolic plane $\mathbf{H}^2$, and for which we choose representatives $h_i \in \mathrm{SL}_2(\mathbb{R})$ with positive eigenvalues. We introduce the embedding

$$j \colon h \in \mathrm{GL}_2(\mathbb{R}) \mapsto \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix} \in \mathrm{PGL}_3(\mathbb{R}) \tag{4.1}$$

and we define $g_i \coloneqq j(h_i) \in \mathrm{PGL}_3(\mathbb{R})$. Each g_i is then a loxodromic element of $\mathrm{PGL}_3(\mathbb{R})$ (having positive eigenvalues) with repulsive and attractive fixed points $p_i^\pm$ parwise distincts on $[e_1, e_2]$ and with $[e_3]$ as a common saddle point (see Example 2.12 for these definitions). In particular, the g_i are in general position and according to Proposition 3.4 there exists for each i some $r_i > 0$, such that $g_1^{r_1}, \dots, g_d^{r_d}$ generates a Schottky subgroup of $\mathrm{PGL}_3(\mathbb{R})$. We replace each h_i by $h_i^{r_i}$, and we

denote $\overline{\Gamma}_0 = \langle \overline{h}_1, \dots, \overline{h}_d \rangle$, $\Gamma_0 = \langle h_1, \dots, h_d \rangle$ (note that $h_i \mapsto \overline{h}_i$ defines an isomorphism from Γ_0 to $\overline{\Gamma}_0$) and
$$\Gamma = j(\Gamma_0) = \langle g_1, \dots, g_d \rangle \subset j(\mathrm{SL}_2(\mathbb{R})).$$
We work from now on with the hyperbolic surface $\Sigma = \overline{\Gamma}_0 \backslash \mathbf{H}^2$ whose geodesic flow on $\mathrm{T}^1\Sigma$ is denoted by (g^t).

4.1. **Hyperbolic surfaces and path structures.** We first define the path structure $\mathcal{L}_\Sigma$ that we will study on $\mathrm{T}^1\Sigma$. Let us recall that a *path structure* on a three-dimensional manifold is a couple $\mathcal{L} = (E^\alpha, E^\beta)$ of one-dimensional distributions whose sum is a contact distribution, and that an *isomorphism* between path structures is a diffeomorphism sending α-distribution on α-distribution, and β-distribution on β-distribution.

4.1.1. *An invariant path structure on* $\mathrm{T}^1\Sigma$. Let us consider on $\mathrm{SL}_2(\mathbb{R})$ the left-invariant one-dimensional distributions E^α and E^β respectively generated by the elements
$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
of its Lie algebra $\mathfrak{sl}_2(\mathbb{R})$. Then $\mathcal{L}_{\mathrm{SL}_2(\mathbb{R})} = (E^\alpha, E^\beta)$ is a left-invariant path structure on $\mathrm{SL}_2(\mathbb{R})$, and the same construction defines on $\mathrm{PSL}_2(\mathbb{R})$ a left-invariant path structure $\mathcal{L}_{\mathrm{PSL}_2(\mathbb{R})}$ for which the two-sheeted covering $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$ is a local isomorphism. We recall that, $\mathrm{PSL}_2(\mathbb{R})$ acting simply transitively on $\mathrm{T}^1\mathbf{H}^2$, we can identify $\mathrm{T}^1\Sigma$ with $\overline{\Gamma}_0 \backslash \mathrm{PSL}_2(\mathbb{R})$. This quotient inherits from $\mathrm{PSL}_2(\mathbb{R})$ a natural path structure and we denote by $\mathcal{L}_\Sigma$ the corresponding structure on $\mathrm{T}^1\Sigma$. If Σ is compact, the geodesic flow is Anosov and the same construction defines a path structure $\mathcal{L}_\Sigma$ whose α (respectively β) direction is the stable (resp. unstable) distribution of the geodesic flow.

Lemma 4.1. *The path structure* $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma)$ *is invariant by the geodesic flow of* Σ.

Proof. We recall first that the geodesic flow of $\mathbf{H}^2$ is conjugated in $\mathrm{PSL}_2(\mathbb{R})$ to the right translations $R_{a^{t/2}}$, where
$$a^t = \begin{bmatrix} \mathrm{e}^t & 0 \\ 0 & \mathrm{e}^{-t} \end{bmatrix},$$
and that the geodesic flow (g^t) of Σ is thus conjugated to $R_{a^{t/2}}$ on $\overline{\Gamma}_0 \backslash \mathrm{PSL}_2(\mathbb{R})$. But the adjoint action of a^t preserves for any t the lines $\mathbb{R}E$ and $\mathbb{R}F$ in $\mathfrak{sl}_2(\mathbb{R})$, and R_{a^t} preserves thus $\mathcal{L}_{\mathrm{SL}_2(\mathbb{R})}$, showing that $\mathcal{L}_\Sigma$ is invariant by the geodesic flow. □

Remark 4.2. It is in fact not difficult to show that, modulo inversion of its distributions E^α and E^β, $\mathcal{L}_{\mathrm{PSL}_2(\mathbb{R})}$ is the only $\mathrm{PSL}_2(\mathbb{R})$-invariant path structure of $\mathrm{PSL}_2(\mathbb{R})$ which is also (R_{a^t})-invariant (see for instance [MM20, Lemme 1.1.14]). In other words, $\mathcal{L}_\Sigma$ is the only path structure of $\mathrm{T}^1\Sigma$ invariant by the geodesic flow that comes from an invariant path structure on $\mathrm{PSL}_2(\mathbb{R})$, and is in this sense the most natural path structure that we could look at on $\mathrm{T}^1\Sigma$.

4.1.2. *A Kleinian path structure.* The algebraic construction that we made has in fact a natural geometrical counterpart. $\mathrm{SL}_2(\mathbb{R})$ can indeed be identified with its only open orbit
$$Y \coloneqq \mathbf{X} \setminus (\mathcal{S}_{\beta,\alpha}[e_1, e_2] \cup \mathcal{S}_{\alpha,\beta}[e_3])$$
in $\mathbf{X}$, which makes of $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma)$ the quotient of Y by a discrete subgroup of $\mathrm{PGL}_3(\mathbb{R})$. This is a particular instance of what is called a *Kleinian* path structure, that is the quotient of an open subset of $\mathbf{X}$ by a discrete subgroup of $\mathrm{PGL}_3(\mathbb{R})$. We introduce the following notations:
$$o' \coloneqq ([1,0,1], [(1,0,1), e_2]) \in \mathbf{X}, g_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = j(-\mathrm{id}), \hat{\Gamma} = \Gamma \cup g_0\Gamma.$$

Lemma 4.3. *1. Y is the $j(\mathrm{SL}_2(\mathbb{R}))$-orbit of o', $j(\mathrm{SL}_2(\mathbb{R}))$ acts simply transitively on Y and $\theta_{o'} \colon h \mapsto h \cdot o'$ is an isomorphism of path structures from $(\mathrm{SL}_2(\mathbb{R}), \mathcal{L}_{\mathrm{SL}_2(\mathbb{R})})$ to $(Y, \mathcal{L}_{\mathbf{X}}|_Y)$.*

2. *$\hat{\Gamma}$ is a discrete subgroup of $j(\mathrm{SL}_2(\mathbb{R}))$.*
3. *$\mathcal{L}_{\mathbf{X}}$ induces a flat path structure on $\hat{\Gamma} \backslash Y$ which is isomorphic to $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma)$.*

Proof. 1. This follows from straightforward calculations (detailed for instance in [MM21, §4.2.2]).
2. Let us assume by contradiction that $\gamma_n \in \hat{\Gamma}$ is a non-stationnary sequence converging to id. Then either some subsequence of (γ_n) is contained in Γ or a subsequence of $(g_0^{-1}\gamma_n)$ does. In both cases this contradicts the discreteness of Γ or the non-stationnary nature of (γ_n), proving that $\hat{\Gamma}$ is indeed discrete.
3. Since $\hat{\Gamma}$ is discrete, it is closed in $j(\mathrm{SL}_2(\mathbb{R}))$, and the action of $\hat{\Gamma}$ is thus free and proper on Y. Now $\theta_{o'}$ defines an isomorphism from $\overline{\Gamma}_0\backslash\mathrm{PSL}_2(\mathbb{R}) \simeq \mathrm{T}^1\Sigma$ to $\hat{\Gamma}\backslash Y$, proving our claim. □

4.2. **A first compactification.** We saw in Proposition 3.7 that for $\gamma_\infty \in \partial_\infty\Gamma$, any subsequence going simply to infinity of the sequence of finite subwords of γ_∞ has balanced dynamics with $B^+_{\alpha\beta}(\gamma_\infty)$ as attractive bouquet of circles. Since $\Gamma \subset j(\mathrm{SL}_2(\mathbb{R}))$, the description of attractive circles in Lemma 2.21 moreover shows that, with $p_+(\gamma_\infty) \in [e_1, e_2]$ the attractive point in $\mathbb{R}\mathbf{P}^2$ of the sequence of finite subwords of γ_∞ (see Lemma 2.11), we have

$$\mathcal{C}^+_\alpha(\gamma_\infty) = \mathcal{C}_\alpha(p_+(\gamma_\infty)) \text{ and } \mathcal{C}^+_\beta(\gamma_\infty) = \mathcal{C}_\beta[e_3, p_+(\gamma_\infty)]. \tag{4.2}$$

Note that $p_+\colon \partial_\infty\Gamma \to [e_1, e_2]$ is an homeomorphism onto its image. According to Proposition 3.6, Γ acts freely, properly and cocompactly on

$$\Omega = \mathbf{X} \setminus \Lambda \text{ with } \Lambda = \bigcup_{\gamma_\infty \in \partial_\infty\Gamma} B^+_{\alpha\beta}(\gamma_\infty).$$

In particular, since $\mathcal{C}_\alpha(p_+(\gamma_\infty)) \subset \mathcal{S}_{\beta,\alpha}[e_1, e_2]$ and $\mathcal{C}_\beta[p_+(\gamma_\infty), e_3] \subset \mathcal{S}_{\alpha,\beta}[e_3]$ we obtain $Y \subset \Omega$, wich directly provides us with a first compactification result.

Proposition 4.4. $\Gamma\backslash\Omega$ *is a path structure compactification of the Kleinian structure* $\Gamma\backslash Y$*, where* $\Gamma\backslash Y$ *embeds as an open and dense subset.*

Proof. The inclusion $Y \subset \Omega$ induces an embedding j of path structures of $\Gamma\backslash Y$ in the closed three-manifold $\Gamma\backslash\Omega$. Moreover, Y being dense in Ω, $j(\Gamma\backslash Y)$ is dense in $\Gamma\backslash\Omega$. □

According to Lemma 4.3, $\Gamma\backslash\Omega$ is a two-sheeted covering of $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma)$, the non-trivial automorphism of this covering being induced by g_0. A naive way to obtain a compactification of $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma)$ should be to take the quotient of $\Gamma\backslash\Omega$ by g_0. But g_0 has a lot of fixed points on Ω: it acts trivially on $\mathcal{C}_\alpha[e_1] \cup \mathcal{C}_\beta[e_1, e_2]$ and on $\mathcal{C} = \{(p, D) \mid p \in [e_1, e_2], D \ni [e_3]\}$, whose intersections with Ω are non-empty. This prevents us from obtaining a smooth quotient of Ω by Γ, and leads us to consider a covering of $\mathbf{X}$ where g_0 will have no fixed points in the preimage of Ω.

4.3. **A journey in a covering of X.** Natural two-sheeted coverings of $\mathbf{X}$ are given by the space $\mathbf{P}(\mathrm{T}\mathbf{S}^2)$ of tangent lines of $\mathbf{S}^2$ and the space $\mathbf{P}^+(\mathrm{T}\mathbb{R}\mathbf{P}^2)$ of tangent half-lines of $\mathbb{R}\mathbf{P}^2$, both endowed with natural actions of $\mathrm{PGL}_3(\mathbb{R})$ and natural path structures given by the pullbacks of $\mathcal{L}_{\mathbf{X}}$. But g_0 acts trivially on the α-circle defined by e_3 in $\mathbf{P}(\mathrm{T}\mathbf{S}^2)$, and on the β-circle defined by (e_1, e_2) in $\mathbf{P}^+(\mathrm{T}\mathbb{R}\mathbf{P}^2)$, whose intersections with the preimage of Ω are non-empty. Hence these coverings are not enough and we have to consider the next one, that is the space $\hat{\mathbf{X}} = \mathbf{P}^+(\mathrm{T}\mathbf{S}^2)$ of tangent half-lines of $\mathbf{S}^2$. We can also think to $\hat{\mathbf{X}}$ as the set of *oriented flags* (d, P) of $\mathbb{R}^3$, d being an oriented line of $\mathbb{R}^3$ contained in an oriented plane P. For (u, v) two non-conlinear vectors of $\mathbb{R}^3$, we will denote by (u, v) the plane $\mathrm{Vect}(u, v)$ oriented by its basis (u, v), and by $(u, (u, v))$ the corresponding point $(\mathbb{R}^+u, (u, v))$ of $\hat{\mathbf{X}}$. Note that $\hat{\mathbf{X}}$ is diffeomorphic to $\mathrm{T}^1\mathbf{S}^2$. In particular, $\hat{\mathbf{X}}$ is orientable and has $\mathbf{S}^3$ as a double-cover. $\hat{\mathbf{X}}$ is a four-sheeted covering of $\mathbf{X}$ through the projection

$$\pi\colon (d, P) \in \hat{\mathbf{X}} \mapsto ([d], [P]) \in \mathbf{X},$$

and we endow $\hat{\mathbf{X}}$ with the path structure $\mathcal{L}_{\hat{\mathbf{X}}} = \pi^*\mathcal{L}_{\mathbf{X}}$. The α and β-leaves of $\mathcal{L}_{\hat{\mathbf{X}}}$ are circles that we denote by $\hat{\mathcal{C}}_\alpha$ and $\hat{\mathcal{C}}_\beta$. For $x \in \mathbf{X}$, $\pi^{-1}(\mathcal{C}_\alpha(x))$ (respectively $\pi^{-1}(\mathcal{C}_\beta(x))$) is the disjoint union of two α-circles in $\hat{\mathbf{X}}$ (resp. of two β-circles), the restriction of π to an α or β-circle of $\hat{\mathbf{X}}$ being a double covering $\mathbf{S}^1 \to \mathbb{R}\mathbf{P}^1$. There is a natural action of $\mathrm{GL}_3(\mathbb{R})$ on $\hat{\mathbf{X}}$ and since the projection $g \mapsto [g]$ of $\mathrm{SL}_3(\mathbb{R})$ in $\mathrm{PGL}_3(\mathbb{R})$ is an isomorphism we can define an action of $\mathrm{PGL}_3(\mathbb{R})$ on $\hat{\mathbf{X}}$ by the formula

$$[g] \cdot x := g \cdot x \text{ for any } g \in \mathrm{SL}_3(\mathbb{R}) \text{ and } x \in \hat{\mathbf{X}}.$$

This action preserve the orientation of $\hat{\mathbf{X}}$ and makes $\pi\colon \hat{\mathbf{X}} \to \mathbf{X}$ equivariant for the respective actions of $\mathrm{PGL}_3(\mathbb{R})$. In particular, $\mathrm{PGL}_3(\mathbb{R})$ preserves $\mathcal{L}_{\hat{\mathbf{X}}}$. Observe that the action of the special orthogonal group $\mathrm{SO}(3)$ is simply transitive on $\hat{\mathbf{X}}$.

To give a better picture of the covering $\hat{\mathbf{X}}$, let us look more closely at the surfaces $\hat{\mathcal{T}}_{\alpha,\beta}(x) = \cup_{y\in\hat{\mathcal{C}}_\alpha(x)}\hat{\mathcal{C}}_\beta(y)$ and $\hat{\mathcal{T}}_{\beta,\alpha}(x) = \cup_{y\in\hat{\mathcal{C}}_\beta(x)}\hat{\mathcal{C}}_\alpha(y)$ for $x\in\hat{\mathbf{X}}$.

Lemma 4.5. *For any $x \in \hat{\mathbf{X}}$, $\hat{\mathcal{T}}_{\alpha,\beta}(x)$ and $\hat{\mathcal{T}}_{\beta,\alpha}(x)$ are tori. Furthermore, $\hat{\mathbf{X}} \setminus \hat{\mathcal{T}}_{\alpha,\beta}(x)$ and $\hat{\mathbf{X}} \setminus \hat{\mathcal{T}}_{\beta,\alpha}(x)$ have two connected components.*

Proof. Since the involution κ defined in (2.13) switches α-β and β-α surfaces, it is sufficient to prove it for $\hat{\mathcal{T}}_{\beta,\alpha}(x)$, and by transitivity of $\mathrm{PGL}_3(\mathbb{R})$, it is sufficient to prove it for $\hat{\mathcal{T}}_{\beta,\alpha}(e_1,e_2) = \left\{(d,P)\in\hat{\mathbf{X}} \;\middle|\; d\in(e_1,e_2)\right\}$. The equality

$$\hat{\mathcal{T}}_{\beta,\alpha}(e_1,e_2) = \bigcup_{(A,B)\in\mathrm{SO}(2)^2}\begin{pmatrix}A & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & B\end{pmatrix}\cdot(e_1,(e_1,e_2))$$

proves that this surface is a torus. Furthermore, $\hat{\mathbf{X}} \setminus \hat{\mathcal{T}}_{\beta,\alpha}(e_1,e_2) = \left\{(d,P)\in\hat{\mathbf{X}} \;\middle|\; d\notin(e_1,e_2)\right\}$ is disconnected since its projection $\hat{\mathbf{S}}^2 \setminus (e_1,e_2)$ on $\hat{\mathbf{S}}^2$ has two connected components C_1 and C_2. Since $\pi^{-1}(C_1)$ and $\pi^{-1}(C_2)$ are both connected (they are in fact solid tori), $\hat{\mathbf{X}} \setminus \hat{\mathcal{T}}_{\beta,\alpha}(e_1,e_2) = \pi^{-1}(C_1)\cup\pi^{-1}(C_2)$ has two connected components. □

The following lemma shows that the subgroup $\hat{\Gamma}$ acts as we wish on $\hat{\Omega} \coloneqq \pi^{-1}(\Omega)$. We point out related results in [ST18, §7.2], where the authors describe cocompact domains of discontinuity for *purely hyperbolic generalized Schottky subgroups* of $\mathrm{PSL}_{2n+1}(\mathbb{R})$ acting on oriented flag spaces.

Lemma 4.6.
1. *Γ acts freely, properly and cocompactly on $\hat{\Omega}$.*
2. *g_0 has no fixed points on $\hat{\mathbf{X}}$.*
3. *$\hat{\Gamma}$ preserves Ω and $\hat{\Omega}$.*
4. *$\hat{\Gamma}$ acts freely and properly on $\hat{\Omega}$.*
5. *$M \coloneqq \hat{\Gamma}\backslash\hat{\Omega}$ is a connected, orientable and closed three-dimensional manifold.*

Proof. 1. Since π is a Γ-equivariant covering, Γ acts as freely and properly on $\hat{\Omega}$ as it does on Ω. The covering $\bar{\pi}\colon \Gamma\backslash\hat{\Omega} \to \Gamma\backslash\Omega$ induced by π having finite fibers and $\Gamma\backslash\Omega$ being compact, $\Gamma\backslash\hat{\Omega}$ is compact as well.
2. The only fixed points of the action of g_0 on $\mathbf{S}^2$ are e_3 and $-e_3$, so that fixed points of g_0 on $\hat{\mathbf{X}}$ are in $\hat{\mathcal{C}}_\alpha(e_3)\cup\hat{\mathcal{C}}_\alpha(-e_3)$. But for $p=e_3$ or $-e_3$, the action of g_0 on $\hat{\mathcal{C}}_\alpha(p)$ is conjugated to the action of $-\mathrm{id}$ on $\mathbf{P}^+(\mathbb{R}^2)$ and has thus no fixed point.
3. We saw in Paragraph 4.2 that the attractive bouquet of any $\gamma_\infty \in \partial_\infty\Gamma$ is of the form $B^+_{\alpha\beta}(\gamma_\infty) = \mathcal{C}_\alpha(p_+)\cup\mathcal{C}_\beta(D_+)$ with $p_+\in[e_1,e_2]$ and $[e_3]\in D_+$. Since g_0 fixes $[e_3]$ and acts trivially on $[e_1,e_2]$ it stabilizes $B^+_{\alpha\beta}(\gamma_\infty)$ and stabilizes thus $\Omega = \mathbf{X}\setminus\bigcup_{\delta_\infty\in\partial_\infty\Gamma}B^+_{\alpha\beta}(\delta_\infty)$ according to Proposition 3.7. Therefore, $\hat{\Gamma}$ stabilizes Ω and thus $\hat{\Omega}$ since π is $\mathrm{PGL}_3(\mathbb{R})$-equivariant.
4. Since Γ acts freely on $\hat{\Omega}$, we only need to show that for any $\gamma\in\Gamma$, $g_0\gamma\neq\mathrm{id}$ has no fixed point on $\hat{\Omega}$ to prove that the action is free. Let us assume by contradiction that $g_0\gamma\cdot x = x$ with $x\in\hat{\Omega}$. Then for any $x\in\mathbb{N}$, $(g_0\gamma)^{2n}\cdot x = x$. But $(g_0\gamma)^{2n} = \gamma^{2n}$ because γ commutes with g_0 which is of order two, hence $\gamma^{2n}\cdot x = x$. Since $\pi(x)\in\Omega$ is outside the repulsive bouquet of circles of (γ^{2n}) (equal to $B^+_{\alpha\beta}(\gamma_\infty)$ with $\gamma_\infty = \gamma^{-2}\gamma^{-2}\gamma^{-2}\cdots\in\partial_\infty\Gamma$), some subsequence of $\gamma^{2n}\cdot\pi(x)$ converges to a point of the attractive bouquet of circles of (γ^{2n}), hence to $\mathbf{X}\setminus\Omega$. This contradicts $\gamma^{2n}\cdot\pi(x) = \pi(\gamma^{2n}\cdot x) = \pi(x)\in\Omega$ and shows that the action is free. Passing to a subsequence and precomposing by g_0^{-1}, for any $\gamma_n\in\hat{\Gamma}$ going to infinity we can assume that (γ_n) is contained in Γ. Now for (x_n) a converging sequence of $\hat{\Omega}$, $(\gamma_n\cdot x_n)$ is not relatively compact by property of the action of Γ on $\hat{\Omega}$, showing that the action of $\hat{\Gamma}$ is proper on $\hat{\Omega}$.
5. Since $\hat{\mathbf{X}}$ is compact and $\hat{\Gamma}\subset\mathrm{PGL}_3(\mathbb{R})$ preserves its orientation, M is orientable and compact as the image of the compact space $\Gamma\backslash\hat{\Omega}$ by the continuous projection $\Gamma\cdot x\mapsto\hat{\Gamma}\cdot x$. □

4.4. **Compactification of the geodesic flow and proof of Theorem A.** We now come back to the path structure $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma)$, and denoting by (g^t) its geodesic flow, we describe the compactification of $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma, g^t)$.

4.4.1. *Geometry of the compactification.* We denote by

$$\Pi \colon \hat{\Omega} \to M = \hat{\Gamma}\backslash\hat{\Omega}$$

the canonical projection, and by $\mathcal{L}$ the path structure of M induced by $\mathcal{L}_{\hat{\mathbf{X}}}$. We introduce

$$\psi^t := j(\mathrm{e}^t \operatorname{id}) = \begin{bmatrix} \mathrm{e}^t & 0 & 0 \\ 0 & \mathrm{e}^t & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This is a flow of unbalanced type β whose repulsive and attractive objects in $\mathbf{X}$ as described in Lemma 2.18 are denoted by $\mathcal{C}^-_\alpha = \mathcal{C}_\alpha[e_3]$, $\mathcal{S}^-_{\alpha,\beta} = \mathcal{S}_{\alpha,\beta}[e_3]$, $\mathcal{C}^+_\beta = \mathcal{C}_\beta[e_1, e_2]$ and $\mathcal{S}^+_{\beta,\alpha} = \mathcal{S}_{\beta,\alpha}[e_1, e_2]$. We also introduce the circle $\mathcal{C} = \{(p, D) \mid p \in [e_1, e_2], D \ni [e_3]\}$ of $\mathbf{X}$. We now define:

$$\mathcal{C}^- = \Pi(\pi^{-1}(\mathcal{C}^-_\alpha \cap \Omega)), \mathcal{T}^- = \Pi(\pi^{-1}(\mathcal{S}^-_{\alpha,\beta} \cap \Omega)), \mathcal{C}^+ = \Pi(\pi^{-1}(\mathcal{C}^+_\beta \cap \Omega)), \mathcal{T}^+ = \Pi(\pi^{-1}(\mathcal{S}^+_{\beta,\alpha} \cap \Omega)),$$
$$\Delta = \Pi(\pi^{-1}(\mathcal{C} \cap \Omega)).$$

Γ acts freely, properly and cocompactly on $[e_1, e_2] \setminus p_+(\partial_\infty\Gamma)$, and $\Gamma\backslash([e_1, e_2] \setminus p_+(\partial_\infty\Gamma))$ is thus a finite union of circles. We denote by $b \in \mathbb{N}^*$ the number of connected components of this quotient, which is the number of boundary components of the topological compact surface with boundary whose interior is homeomorphic to Σ, that is the number of funnels of the hyperbolic surface Σ (considering the case of $d = 1$ generators, that is the case where Σ is a hyperbolic cylinder, can be useful to understand these equalities).

Proposition 4.7. *1. $\mathcal{C}^-$ (respectively $\mathcal{C}^+$, respectively Δ) is the disjoint union of b pairs of disjoint circles $\{\mathcal{C}^-_i\}_{i=1}^b$ (resp. $\mathcal{C}^+_i$, resp. Δ_i). $\mathcal{T}^-$ (resp. $\mathcal{T}^+$) is the disjoint union of b tori $\{\mathcal{T}^-_i\}_{i=1}^b$ (resp. $\mathcal{T}^+_i$). Furthermore $\mathcal{C}^-$, $\mathcal{C}^+$ and Δ are pairwise disjoint, $\mathcal{C}^-_i \subset \mathcal{T}^-_i$, $\mathcal{C}^+_i \subset \mathcal{T}^+_i$ and $\mathcal{T}^-_i \cap \mathcal{T}^+_i = \Delta_i$ for each i, and $\mathcal{T}^-_i \cap \mathcal{T}^+_j = \varnothing$ for any $i \neq j$.*

2. There exists in $(M, \mathcal{L})$ four disjoint open sets $\{N_j\}_{j=1}^4$ isomorphic to $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma)$. Furthermore, $M \setminus \sqcup_{j=1}^4 N_j = \mathcal{T}^- \cup \mathcal{T}^+$.

3. The flow (ψ^t) defines on M a flow (φ^t) of automorphisms of $\mathcal{L}$, conjugated on each of the N_j to (g^{2t}), where (g^t) denotes the geodesic flow of $\mathrm{T}^1\Sigma$.

We emphasize that (φ^t) is conjugated to (g^{2t}) and not to (g^t). We will however consider (φ^t) rather than $(\varphi^{\frac{t}{2}})$ which would be quite inconvenient.

Proof or Proposition 4.7. 1. We first emphasize that $\mathcal{C}^-_\alpha$, $\mathcal{S}^-_{\alpha,\beta}$, $\mathcal{C}^+_\beta$, $\mathcal{S}^+_{\beta,\alpha}$ and $\mathcal{C}$ are $\mathrm{Stab}[e_3] \cap \mathrm{Stab}[e_1, e_2] = j(\mathrm{GL}_2(\mathbb{R}))$-invariant, and thus $\hat{\Gamma}$-invariant. Now, π being $\mathrm{PGL}_3(\mathbb{R})$-equivariant and $\hat{\Omega}$ being $\hat{\Gamma}$-invariant, $\pi^{-1}(\mathcal{C}^-_\alpha \cap \Omega)$, $\pi^{-1}(\mathcal{S}^-_{\alpha,\beta} \cap \Omega)$, $\pi^{-1}(\mathcal{C}^+_\beta \cap \Omega)$, $\pi^{-1}(\mathcal{S}^+_{\beta,\alpha} \cap \Omega)$ and $\pi^{-1}(\mathcal{C} \cap \Omega)$ are $\hat{\Gamma}$-invariant. These are closed subsets of Ω, and their projections by Π are thus closed in M, hence compact, which already proves that $\mathcal{C}^-$, $\mathcal{C}^+$ and Δ are finite unions of circles. For any connected component I_k of $\mathcal{C}^+_\beta \cap \Omega$, $\pi^{-1}(I_k)$ has four connected components and g_0 preserves $\pi^{-1}(I_k)$ and has two orbits on its space of connected components. Since $\hat{\Gamma} = \langle \Gamma, g_0 \rangle$, this shows that the space of connected components of $\mathcal{C}^+ = \hat{\Gamma}\backslash\pi^{-1}(\mathcal{C}^+_\beta \cap \Omega)$ surjects with a fiber of cardinal two onto the one of $\Gamma\backslash(\mathcal{C}^+_\beta \cap \Omega)$. The same happens between $\mathcal{C}^+$ (respectively Δ) and $\Gamma\backslash(\mathcal{C}^+_\beta \cap \Omega)$ (resp. $\Gamma\backslash(\mathcal{C} \cap \Omega)$). But $\Gamma\backslash(\mathcal{C}^-_\alpha \cap \Omega)$, $\Gamma\backslash(\mathcal{C}^+_\beta \cap \Omega)$ and $\Gamma\backslash(\mathcal{C} \cap \Omega)$ have the same number of connected components than $\Gamma\backslash([e_1, e_2] \setminus p_+(\partial_\infty\Gamma))$, that is b, which proves the claim concerning $\mathcal{C}^-$, $\mathcal{C}^+$ and Δ.

Since the compact surfaces $\mathcal{T}^-$ and $\mathcal{T}^+$ bear smooth one-dimensional distributions, they have Euler characteristic equal to zero according to Poincaré-Hopf Theorem, and we only have to check that they are indeed orientable to prove that they are finite unions of tori. We saw in Paragraph 4.2 that $\Lambda = \cup_{\gamma_\infty \in \partial_\infty\Gamma} \mathcal{C}_\alpha(p_+(\gamma_\infty)) \cup \mathcal{C}_\beta[p_+(\gamma_\infty), e_3]$ where $p_+(\gamma_\infty) \in [e_1, e_2]$, and thus $\mathcal{C}^-_\alpha \cap \Omega = \cup_k I_k$ with $\{I_k\}$ a collection of disjoint intervals in the circle $\mathcal{C}^-_\alpha$. Therefore $\mathcal{S}^-_{\alpha,\beta} \cap \Omega = \cup_k \cup_{x \in I_k} \mathcal{C}_\beta[x]$ is a union of cylinders, is thus orientable, and $\pi^{-1}(\mathcal{S}^-_{\alpha,\beta}\cap\Omega)$ is orientable as well. Since the action of $\mathrm{PGL}_3(\mathbb{R})$ preserves the orientation of these cylinders, their projections in M are orientable compact surfaces of Euler characteristic zero, that is tori. Finally $\mathcal{T}^-$ is a finite union of tori, and the same holds for $\mathcal{T}^+$ for the same reasons. The fact that the number of connected components of $\mathcal{T}^-$ (respectively $\mathcal{T}^+$) is half of the one of $\mathcal{C}^-$ (resp. $\mathcal{C}^+$) is deduced from the fact that for any $x\in\mathbf{X}$, the preimage of $\mathcal{S}_{\alpha,\beta}(x)$ (resp. $\mathcal{S}_{\beta,\alpha}(x)$) in $\hat{\mathbf{X}}$ is connected, whereas the preimage of $\mathcal{C}_\alpha(x)$ (resp. $\mathcal{C}_\beta(x)$) has two connected components.

The last claim directly follows from the fact that $\mathcal{C}^-_\alpha$, $\mathcal{C}^+_\beta$ and $\mathcal{C}$ are pairwise disjoint, and that $\mathcal{S}^-_{\alpha,\beta}\cap\mathcal{S}^+_{\beta,\alpha}=\mathcal{C}$.
2. We recall that $o'=([1,0,1],[(1,0,1),e_2])\in Y$ and we denote $\pi^{-1}(o')=\{\hat{o}_i\}_{i=1,\dots,4}$. For any $i\neq k$, $\hat{o}_i$ and $\hat{o}_k$ are not in the same $j(\mathrm{SL}_2(\mathbb{R}))$-orbit. Since $j(\mathrm{SL}_2(\mathbb{R}))$ acts freely at $\hat{o}_i$, the formula $\iota_i(g\cdot o')=g\cdot\hat{o}_i$ for any $g\in j(\mathrm{SL}_2(\mathbb{R}))$ defines a map $\iota_i\colon Y\to\hat{\Omega}$ descending for each $i=1,\dots,4$ to an embedding $\bar{\iota}_i$ of $\hat{\Gamma}\backslash Y\simeq\mathrm{T}^1\Sigma$ in the compact path structure M. The images $N_i=\bar{\iota}_i(\hat{\Gamma}\backslash Y)$ of these embeddings are disjoint as projections in M of distinct orbits of $j(\mathrm{SL}_2(\mathbb{R}))$ and the equality $M\setminus\cup_i N_i=\mathcal{T}^-\cup\mathcal{T}^+$ directly follows from $\mathbf{X}\setminus Y=\mathcal{S}^-_{\alpha,\beta}\cup\mathcal{S}^+_{\beta,\alpha}$.
3. We saw in Lemma 4.1 that the geodesic flow of Σ is conjugated to $(R_{a^{t/2}})$ on $\bar{\Gamma}_0\backslash\mathrm{PSL}_2(\mathbb{R})$, and the relation $j(ga^t)\cdot o'=j(e^t\,\mathrm{id})\cdot(j(g)\cdot o')$ for any $t\in\mathbb{R}$ and $g\in\mathrm{SL}_2(\mathbb{R})$ shows that (R_{a^t}) is itself conjugated in Y to (ψ^t). Since (ψ^t) acts trivially on $[e_1,e_2]$ and fixes $[e_3]$, it preserves Ω, and hence $\hat{\Omega}$. Since (ψ^t) commutes with g_0 and all the g_i, it descends to a flow of automorphisms of M that we denote by (φ^t), conjugated to (g^{2t}) on each N_i. □

4.4.2. *Dynamics at infinity of the geodesic flow.* We now describe the dynamics of (φ^t) on M. The set of fixed points of (ψ^t) on $\mathbf{S}^2$ being $\{e_3\}\cup(e_1,e_2)$, $\pi^{-1}(\mathcal{C}^-_\alpha\cup\mathcal{C}^+_\beta\cup\mathcal{C})$ is the set of fixed points of (ψ^t) on $\hat{\mathbf{X}}$, and each point of $\mathcal{C}^-\cup\mathcal{C}^+\cup\Delta$ is thus a fixed point of (φ^t).

Lemma 4.8. *The set of fixed points of (φ^t) is precisely $\mathcal{C}^-\cup\mathcal{C}^+\cup\Delta$.*

Proof. 1. Let $x\in\hat{\Omega}$ such that $\Pi(x)$ is a fixed point of (φ^t). For any $t\in\mathbb{R}$ there exists then $\gamma^t\in\hat{\Gamma}$ such that $\psi^t(x)=\gamma^t(x)$, and such a γ^t is unique since $\hat{\Gamma}$ acs freely on $\hat{\Omega}$. Moreover, $\gamma^{s+t}(x)=\psi^s(\gamma^t(x))=\gamma^t(\psi^s(x))=\gamma^t\gamma^s(x)$ since $\hat{\Gamma}$ and (ψ^t) commute, hence $\gamma^{s+t}=\gamma^s\gamma^t$. Finally (γ^t) is a one-parameter subgroup of $\hat{\Gamma}$, implying $\gamma^t=\mathrm{id}$ for any $t\in\mathbb{R}$ since $\hat{\Gamma}$ is discrete. Therefore x is a fixed point of (ψ^t), that is $x\in\pi^{-1}(\mathcal{C}^-_\alpha\cup\mathcal{C}^+_\beta\cup\mathcal{C})\cap\hat{\Omega}$, which proves our claim. □

The dynamics of the flow (ψ^t) described in Lemma 2.18 allow us to obtain an accurate picture of those of (φ^t). We denote by $\phi_+\colon\mathbf{X}\to\mathcal{C}^+_\beta$ the application associated in Lemma 2.18 to the flow (ψ^t) of unbalanced type β, and by $\phi_-\colon\mathbf{X}\to\mathcal{C}^-_\alpha$ the application associated to its inverse (ψ^{-t}) of unbalanced type α in Lemma 2.17.

Proposition 4.9. *We introduce $\Upsilon^-=\Pi(\pi^{-1}(\phi_+^{-1}(\Lambda)\cap\Omega))$ and $\Upsilon^+=\Pi(\pi^{-1}(\phi_-^{-1}(\Lambda)\cap\Omega))$.*

1. *Υ^- and Υ^+ are contained in the union $\cup_{i=1}^4 N_i$ of the copies of $\mathrm{T}^1\Sigma$ in M.*
2. *The closure of Υ^- (respectively Υ^+) is equal to $\Upsilon^-\cup\mathcal{C}^-$ (resp. $\Upsilon^+\cup\mathcal{C}^+$), and has empty interior. In particular, $M\setminus(\mathcal{T}^-\cup\Upsilon^-)$ and $M\setminus(\mathcal{T}^+\cup\Upsilon^+)$ are dense and open subsets of M.*
3. *There exists two continuous applications*
$$\Phi_+\colon M\setminus(\mathcal{T}^-\cup\Upsilon^-)\to\mathcal{C}^+ \text{ and } \Phi_-\colon M\setminus(\mathcal{T}^+\cup\Upsilon^+)\to\mathcal{C}^-$$
such that $\mathcal{D}_{(\varphi^t)}(x)=\Phi_+(x)$ for any $x\in M\setminus(\mathcal{T}^-\cup\Upsilon^-)$, and $\mathcal{D}_{(\varphi^{-t})}(x)=\Phi_-(x)$ for any $x\in M\setminus(\mathcal{T}^+\cup\Upsilon^+)$.
4. *For any $1\leq i\leq 4$, $\Upsilon^-\cap N_i$ (respectively $\Upsilon^+\cap N_i$) is (the image of) the subset of points of $\mathrm{T}^1\Sigma$ whose ω-limit set (resp. α-limit set) for the geodesic flow is non-empty.*
5. *Let $K\subset M\setminus(\mathcal{T}^-\cup\Upsilon^-)$ (respectively $K\subset M\setminus(\mathcal{T}^+\cup\Upsilon^+)$) be a compact subset and $t_n\to+\infty$ (resp. $t_n\to-\infty$) such that $\varphi^{t_n}(K)$ converges. Then $\lim\varphi^{t_n}(K)\subset\Phi_+(K)$ (resp. $\lim\varphi^{t_n}(K)\subset\Phi_-(K)$). If K is moreover the closure of its interior, then*
$$\lim_{t\to+\infty}\varphi^t(K)=\Phi_+(K)$$

(resp. $\lim_{-\infty} \varphi^t(K) = \Phi_-(K)$*).*

We recall the definition of the *ω-limit set*

$$\omega(x) = \left\{\text{accumulation points of } g^{t_n}(x) \;\middle|\; t_n \to +\infty\right\}$$

of x for g^t, its α-limit set being the corresponding subset for the sequences $t_n \to -\infty$.

Proof of Proposition 4.9. The arguments are formally the same in the past and in the future, that is concerning (φ^t) and Υ^-, and concerning (φ^{-t}) and Υ^+. We thus only write them for (ψ^t).
1. and 2. Since g_0 acts trivially on $[e_1, e_2]$, the Γ-invariant set Λ is actually $\hat{\Gamma}$-invariant. We saw in Lemma 2.18 that ϕ_+ is equivariant with respect to a morphism $\rho_\infty \colon \mathrm{Stab}[e_3] \to \mathrm{Stab}[e_3] \cap \mathrm{Stab}[e_1, e_2]$, and the construction of ρ_∞ in (2.7) shows that ρ_∞ is equal to the identity in restriction to $j(\mathrm{GL}_2(\mathbb{R}))$ and thus in restriction to $\hat{\Gamma}$. Hence ϕ_+ is $\hat{\Gamma}$-equivariant, and $\phi_+^{-1}(\Lambda)$ is $\hat{\Gamma}$-invariant. Now the description of ϕ_+ in the proof of Lemma 2.18 and the description of Ω in Paragraph 4.2 (see (4.2)) shows that

$$\phi_+^{-1}(\Lambda) \cap \Omega = \bigcup_{\gamma_\infty \in \partial_\infty \Gamma} \mathcal{S}_{\beta,\alpha}[p_+(\gamma_\infty), e_3] \setminus \Big(\mathcal{C}_\alpha[e_3] \cup \mathcal{C}_\beta[p_+(\gamma_\infty), e_3] \cup \mathcal{C}_\alpha(p_+(\gamma_\infty))\Big).$$

In particular, $\phi_+^{-1}(\Lambda) \cap \Omega$ is disjoint from $\mathcal{S}^+_{\beta,\alpha}$ and $\mathcal{S}^-_{\alpha,\beta}$ and Υ^- is thus disjoint from $\mathcal{T}^+$ and $\mathcal{T}^-$, hence contained in $\cup_{i=1}^4 N_i$. Furthermore $\pi^{-1}(\phi_+^{-1}(\Lambda) \cap \Omega)$ is a $\hat{\Gamma}$-invariant subset of empty interior, therefore Υ^- has empty interior. Since ϕ_+ is not continuous on Ω, $\phi_+^{-1}(\Lambda) \cap \Omega$ is not closed in Ω. However, ϕ_+ being continous on $\mathbf{X} \setminus \mathcal{C}^-_\alpha$, $\phi_+^{-1}(\Lambda) \cap \Omega$ is closed in $\Omega \setminus \mathcal{C}^-_\alpha$. Hence $\Upsilon^- \setminus \mathcal{C}^-$ is closed in $M \setminus \mathcal{C}^-$, and the closure of Υ^- is contained in $\Upsilon^- \cup \mathcal{C}^-$. In particular, $\mathrm{Cl}(\Upsilon^-)$ has empty interior. More precisely, let $\gamma_\infty \in \partial_\infty \Gamma$, p_n a sequence of $[p_+(\delta_\infty), e_3]$ converging to $[e_3]$, and $p \in [e_1, e_2] \setminus p_+(\partial_\infty \Gamma)$. Then $(p_n, [p_n, p]) \in \phi_+^{-1}(\Lambda) \cap \Omega$ converges to $([e_3], [e_3, p]) \in \mathcal{C}^-_\alpha \cap \Omega$. This shows not only that the closure of Υ^- is equal to $\Upsilon^- \cup \mathcal{C}^-$, but that any connected component of Υ^- accumulate on one of the connected components of $\mathcal{C}^-$. In particular $\mathcal{T}^- \cup \Upsilon^- = \mathcal{T}^- \cup \mathrm{Cl}(\Upsilon^-)$ is a closed subset with empty interior, and $M \setminus (\mathcal{T}^- \cup \Upsilon^-)$ is an open and dense subset.
3. Let $x \in M \setminus (\mathcal{T}^- \cup \Upsilon^-)$, and let $x_n \in M$ converging to x and $t_n \in \mathbb{R}$ to $+\infty$, such that $\lim \varphi^{t_n}(x_n) = x_\infty \in \mathcal{D}_{(\varphi^t)}(x)$. We choose $\hat{x} \in \Pi^{-1}(x)$, and there exists a sequence $\hat{x}_n \in \Pi^{-1}(x_n)$ converging to $\hat{x}$. Passing to a subsequence, we can furthermore assume that $\lim \psi^{t_n}(\hat{x}_n) = \hat{x}_\infty \in \hat{\mathbf{X}}$ by compacity of $\hat{\mathbf{X}}$. Then $\bar{x}_n = \pi(\hat{x}_n)$ converges to $\bar{x} = \pi(\hat{x}) \notin \mathcal{S}^-_{\alpha,\beta}$ and $\psi^{t_n}(\bar{x}_n)$ to $\bar{x}_\infty = \pi(\hat{x}_\infty)$. According to Lemma 2.18 $\bar{x}_\infty = \phi_+(\bar{x}) \in \mathcal{C}^+_\beta$ and since $x \notin \Upsilon^-$, $\bar{x}_\infty \in \Omega$ and we thus have $\hat{x}_\infty \in \pi^{-1}(\phi_+(\bar{x})) \subset \hat{\Omega}$. In particular $\hat{x}_\infty \notin \hat{\mathcal{T}}_{\alpha,\beta}(e_3)$. Since $x \notin \mathcal{T}^-$, $\hat{x} \in \hat{\mathbf{X}} \setminus \hat{\mathcal{T}}_{\alpha,\beta}(e_3)$ which has two connected components according to Lemma 4.5. Since $((e_1), (e_1, e_2))$ and $((e_1), (e_1, -e_2))$ are not in the same connected component of $\hat{\mathbf{X}} \setminus \hat{\mathcal{T}}_{\alpha,\beta}(e_3)$ and are fixed by (ψ^t), each of these components is preserved by (ψ^t). We denote by C the connected component containing $\hat{x}$. For n large enough, $\hat{x}_n \in C$ and thus $\psi^{t_n}(\hat{x}_n) \in C$, showing that $\hat{x}_\infty \in C$. We already saw that $\hat{x}_\infty \in \pi^{-1}(\phi_+(\bar{x}))$, and $C \cap \pi^{-1}(\phi_+(\bar{x}))$ has cardinal two: if $\phi_+(\bar{x}) = (p, [e_1, e_2])$, then $C \cap \pi^{-1}(\phi_+(\bar{x})) = \{(\pm p, \varepsilon(e_1, e_2))\}$ with ε the orientation corresponding to the connected component C. Since $g_0 \in \hat{\Gamma}$ identifies the two points $(\pm p, \varepsilon(e_1, e_2))$, $\Pi(C \cap \pi^{-1}(\phi_+(\bar{x})))$ is a point of $\mathcal{C}^+$ depending only on x, that we denote by $\Phi_+(x)$. We have shown that $\mathcal{D}_{(\varphi^t)}(x) \subset \{\Phi_+(x)\}$, but $\mathcal{D}_{(\varphi^t)}(x) \neq \varnothing$ since M is compact, and this inclusion is thus an equality. The continuity of Φ_+ on $M \setminus \mathcal{C}^-$ follows from the one of ϕ_+ on $\mathbf{X} \setminus \mathcal{C}^-_\alpha$, proved in Lemma 2.18.
4. Let $x \in \mathrm{T}^1\Sigma$ whose ω-limit set is non-empty, and y be the corresponding point in one of the copies N_i, with respect to an isomorphism conjugating (φ^t) with the geodesic flow. Then if $y \notin \Upsilon^-$ by contradiction, the ω-limit set of y for (φ^t) would be disjoint from N_i according to the previous claim, and the ω-limit set of x for the geodesic flow would thus be empty. Conversely, let $x \in \Upsilon^-$ contained in the copy N_i of $\mathrm{T}^1\Sigma$, and y be the corresponding point of $\mathrm{T}^1\Sigma$. Let $t_n \to +\infty$ such that $\lim \varphi^{t_n}(x) = x_\infty$ in the ω-limit set of x for (φ^t). With $\hat{x} \in \Pi^{-1}(x)$ and $\bar{x} = \pi(\hat{x})$, passing to a subsequence we can assume that $\psi^{t_n}(\bar{x})$ converges in $\mathbf{X}$, and then $\lim \psi^{t_n}(\bar{x}) = \phi_+(\bar{x}) \in \Lambda$ by hypothesis. By cocompacity of the action of Γ on Ω, there exists a sequence $\gamma_n \in \Gamma$ such that $\gamma_n \psi^{t_n}(\bar{x})$ is relatively compact in Ω, and we can assume that $\gamma_n \psi^{t_n}(\bar{x})$ converges to $\bar{x}_\infty \in \Omega$,

possibly taking a new subsequence. Since $\phi_+^{-1}(\Lambda)\cap\Omega$ is invariant by Γ and by (φ^t), $\gamma_n\psi^{t_n}(\bar{x})\in\phi_+^{-1}(\Lambda)\cap\Omega$, and $\bar{x}_\infty\in\phi_+^{-1}(\Lambda)\cap\Omega\cup\mathcal{C}_\alpha^-$ since ϕ_+ is continuous on $\mathbf{X}\setminus\mathcal{C}_\alpha^-$. Let us temporarily assume that $\bar{x}_\infty\notin\mathcal{C}_\alpha^-$, which will be proved thereafter. Then $x_\infty\in\Upsilon$, which shows that the ω-limit set of y for (φ^t) is contained in Υ, and thus in N_i. Therefore the ω-limit set of x for the geodesic flow is non-empty, finishing the proof

It only remains to prove that $\bar{x}_\infty\notin\mathcal{C}_\alpha^-$. Since $\psi^{t_n}(\bar{x})$ goes to infinity in Ω, γ_n goes to infinity in Γ, and passing to a subsequence we can assume that γ_n goes simply to infinity. According to Proposition 3.7, γ_n converges then to a point $\gamma_\infty\in\partial_\infty\Gamma$, is of balanced type, and has $B^+_{\alpha\beta}(\gamma_\infty)$ as attractive bouquet of circles. Since $\bar{x}_\infty=\lim\gamma_n\psi^{t_n}(\bar{x})\notin\Lambda$, in particular $\bar{x}_\infty\notin B^+_{\alpha\beta}(\gamma_n)$, which implies $\lim\psi^{t_n}(\bar{x})=\phi_+(\bar{x})\in B^-_{\alpha\beta}(\gamma_n)$ according to Lemma 2.21. Since the repulsive point $x_-(\gamma_n)$ of (γ_n) is of the form $(p_+(\delta_\infty),[p_+(\delta_\infty),e_3])$ for some $\delta_\infty\in\partial_\infty\Gamma$, we have more precisely $\lim\psi^{t_n}(\bar{x})\in\mathcal{C}_\alpha^-(\gamma_n)\setminus\{x_-(\gamma_n)\}$ and thus $\bar{x}_\infty\in\mathcal{S}^+_{\alpha,\beta}(\gamma_n)$ according to Lemma 2.21 again. Since $\bar{x}_\infty\in\Omega$ and $\mathcal{S}^+_{\alpha,\beta}(\gamma_n)\cap\mathcal{C}_\alpha^-=([e_3],[e_3,p_+(\gamma_n)])\in\mathcal{C}^+_\beta(\gamma_n)\subset\Lambda$, this shows that $\bar{x}_\infty\notin\mathcal{C}_\alpha^-$ and concludes the proof.

5. Let $t_n\in\mathbb{R}$ be a sequence such that $t_n\to+\infty$. According to the third claim of this proposition $\cup_{x\in K}\mathcal{D}_{(\varphi^{t_n})}(x)=\Phi_+(K)$, which proves that $\lim\varphi^{t_n}(K)\subset\Phi_+(K)$ according to Lemma 2.2. If moreover $K=\mathrm{Cl}(\mathrm{Int}\,K)$, then since $\cup_{x\in\mathrm{Int}\,K}\mathcal{D}_{(\varphi^{t_n})}(x)=\Phi_+(\mathrm{Int}\,K)$ according to the third claim of the Proposition, we get $\mathrm{Cl}(\cup_{x\in\mathrm{Int}\,K}\mathcal{D}_{(\varphi^{t_n})}(x))=\Phi_+(K)$ $(=\cup_{x\in K}\mathcal{D}_{(\varphi^{t_n})}(x))$ by continuity of Φ_+, implying that $\lim\varphi^{t_n}(K)=\Phi_+(K)$ according to Lemma 2.2. Finally $\lim\limits_{t\to+\infty}\varphi^t(K)=\Phi_+(K)$ since the latter was proved for any sequence $t_n\to+\infty$. □

4.4.3. *Invariant probability measures.* Endowing M with a Riemannian metric, for any $x\in M$, $u\in\mathrm{T}_xM\setminus\{0\}$ and $t\in\mathbb{R}^*$ we denote

$$\lambda_t(x,u)=\frac{1}{t}\ln\left\|\mathrm{D}_x\varphi^t(u)\right\|.$$

Then for $\varepsilon=\pm1$, the existence and the value (if it exists) of the limit

$$\lambda_\varepsilon(x,u)\coloneqq\lim_{t\to\varepsilon\infty}\lambda_t(x,u)$$

is independent of the chosen Riemannian metric since M is compact. In the rest of this paragraph, the sentence "$\lambda_\varepsilon(x,u)=\lambda$" will mean: "$\lambda_\varepsilon(x,u)$ exists and is equal to λ". Note that if $\lambda_\varepsilon(x,u)$ exists, then for any $t\in\mathbb{R}$: $\lambda_\varepsilon(\varphi^t(x),\mathrm{D}_x\varphi^t(u))=\lambda_\varepsilon(x,u)$. We denote $\mathcal{L}=(E^\alpha,E^\beta)$ and $E^c=\mathbb{R}\frac{d\varphi^t}{dt}$ the direction of the flow on $M\setminus(\mathcal{C}^-\cup\Delta\cup\mathcal{C}^+)$.

Proposition 4.10. *For any $x\in M\setminus(\mathcal{T}^-\cup\Upsilon^-\cup\mathcal{T}^+\cup\Upsilon^+)$:*

1. *$\forall u\in\mathrm{T}_xM\setminus(E^c(x)\oplus E^\alpha(x))$: $\lambda_+(x,u)=0$;*
2. *$\forall u\in E^c(x)\oplus E^\alpha(x)\setminus\{0\}$: $\lambda_+(x,u)=-1$;*
3. *$\forall u\in\mathrm{T}_xM\setminus(E^c(x)\oplus E^\beta(x))$: $\lambda_-(x,u)=0$;*
4. *$\forall u\in E^c(x)\oplus E^\beta(x)\setminus\{0\}$: $\lambda_-(x,u)=1$.*

Proof. Let $x\in M\setminus(\mathcal{T}^-\cup\Upsilon^-\cup\mathcal{T}^+\cup\Upsilon^+)$ and $\hat{x}\in\Pi^{-1}(x)$. We first prove the two claims concerning positive times. According to the proof of the third part of Proposition 4.9 there exists $\hat{x}_\infty\in\Pi^{-1}(\Phi_+(x))$ such that $\lim\limits_{t\to+\infty}\psi^t(\hat{x})=\hat{x}_\infty$. Denoting by $\|\cdot\|'$ the pullback of $\|\cdot\|$ on $\hat{\Omega}$, with $\varepsilon=\alpha$ or β we have $\lim\left\|\mathrm{D}_x\varphi^t|_{E^\varepsilon}\right\|=\lim\left\|\mathrm{D}_{\hat{x}}\psi^t|_{\hat{\mathcal{E}}_\varepsilon}\right\|'$. Furthermore for any $g\in j(\mathrm{GL}_2(\mathbb{R}))$, denoting by $\|\cdot\|''$ the pushforward of $\|\cdot\|'$ by g on $g(\hat{\Omega})$ we have $\lim\left\|\mathrm{D}_{\hat{x}}\psi^t|_{\hat{\mathcal{E}}_\varepsilon}\right\|'=\lim\left\|\mathrm{D}_{g(\hat{x})}g\psi^tg^{-1}|_{\hat{\mathcal{E}}_\varepsilon}\right\|''=\lim\left\|\mathrm{D}_{g(\hat{x})}\psi^t|_{\hat{\mathcal{E}}_\varepsilon}\right\|''$, because $j(\mathrm{GL}_2(\mathbb{R}))$ centralizes (ψ^t). We can thus assume that $\hat{x}_\infty=((e_1),(e_1,e_2))$ and make the calculations for (ψ^t) and a Riemannian metric defined around $\hat{x}_\infty$. The claim that we want to prove being (ψ^t)-invariant, we can moreover assume that $\hat{x}$ is as close to $\hat{x}_\infty$ as we want, and we will assume that $\hat{x}=\phi_1(p)$ with $p\in\mathbb{R}^3$ and $\phi_1\colon\mathbb{R}^3\to\hat{\mathbf{X}}$ the chart defined around $\hat{x}_\infty=\phi_1(0,0,0)$ by

$$\phi_1(x,y,z)=\left(\left(\begin{smallmatrix}1\\x\\y\end{smallmatrix}\right),\left(\left(\begin{smallmatrix}1\\x\\y\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\\z\end{smallmatrix}\right)\right)\right).\tag{4.3}$$

In this chart (ψ^t) is linearized: $A^t \coloneqq \phi_1^{-1}\circ\psi^t\circ\phi_1$ is the diagonal linear flow $\mathrm{Diag}(1,\mathrm{e}^{-t},\mathrm{e}^{-t})$, and our claim is now reduced to the asymptotical study of A^t in the neighbourhood of $(0,0,0)$, for any Riemannian metric $\|\cdot\|$ defined around $(0,0,0)$ (the claim being independent of this metric since we stay in a compact set for positive times). But the asymptotics of (A^t) are simply given by its eigenspaces and eigenvalues:

- for any $u\in\mathbb{R}^3\setminus\mathrm{Vect}(e_2,e_3)$: $\lim\limits_{t\to+\infty}\frac{1}{t}\ln\|A^t(u)\|=0$;
- for any $u\in\mathrm{Vect}(e_2,e_3)\setminus\{0\}$: $\lim\limits_{t\to+\infty}\frac{1}{t}\ln\|A^t(u)\|=-1$.

We are thus left to check that $\mathbb{R}\frac{dA^t(p)}{dt}|_{t=0}\oplus\phi_1^*\hat{\mathcal{E}}_\alpha(p)=\mathrm{Vect}(e_2,e_3)$ to conclude the proof of the claims 1 and 2. Note that $\phi_1^*\hat{\mathcal{E}}_\alpha$ is constant equal to $\mathbb{R}e_3$ and $\frac{dA^t(x,y,z)}{dt}|_{t=0}=(0,-y,-z)$. Denoting $p=(x_0,y_0,z_0)\in\mathbb{R}^3$, since $x\notin\mathcal{T}^+$ by hypothesis, $\phi_1(p)=\hat{x}\notin\hat{\mathcal{T}}_{\beta,\alpha}(e_1,e_2)$ which means that $y_0\neq0$. This implies that $\mathbb{R}\frac{dA^t(p)}{dt}|_{t=0}\oplus\phi_1^*\hat{\mathcal{E}}_\alpha(p)=\mathrm{Vect}(e_2,e_3)$ as we wanted and conclude the proof for positive times.

For negative times, in order to linearize (φ^t) we can do the same preliminary reductions and assume that $\lim\limits_{t\to-\infty}\psi^t(\hat{x})=((e_3),(e_3,e_2))$ with $\hat{x}\in\Pi^{-1}(x)$ and that $\hat{x}=\phi_2(p)$ with $p\in\mathbb{R}^3$ and $\phi_2\colon\mathbb{R}^3\to\hat{\mathbf{X}}$ the chart defined around $((e_3),(e_3,e_2))=\phi_2(0,0,0)$ by

$$\phi_2(x,y,z)=\left(\left(\begin{smallmatrix}x\\y\\1\end{smallmatrix}\right),\left(\left(\begin{smallmatrix}x\\y\\1\end{smallmatrix}\right),\left(\begin{smallmatrix}z\\1\\0\end{smallmatrix}\right)\right)\right). \tag{4.4}$$

Then $B^t\coloneqq\phi_2^{-1}\circ\psi^t\circ\phi_2$ is the linear diagonal flow $\mathrm{Diag}(\mathrm{e}^t,\mathrm{e}^t,1)$, and as before:

- for any $u\in\mathbb{R}^3\setminus\mathrm{Vect}(e_1,e_2)$: $\lim\limits_{t\to-\infty}\frac{1}{t}\ln\|B^t(u)\|=0$;
- for any $u\in\mathrm{Vect}(e_1,e_2)\setminus\{0\}$: $\lim\limits_{t\to-\infty}\frac{1}{t}\ln\|B^t(u)\|=1$.

Denoting $p=(x_0,y_0,z_0)\in\mathbb{R}^3$, since $x\notin\mathcal{T}^-$ by hypothesis, $\phi_2(p)\notin\hat{\mathcal{T}}_{\alpha,\beta}(e_3)$ which means that the determinant $x_0-y_0z_0$ is non-zero. Since $\phi_2^*\hat{\mathcal{E}}_\alpha(x,y,z)=\mathbb{R}(ze_1+e_2)$ and $\frac{dB^t(x,y,z)}{dt}|_{t=0}=(x,y,0)$, this proves that $\mathbb{R}\frac{dB^t(p)}{dt}|_{t=0}\oplus\phi_1^*\hat{\mathcal{E}}_\beta(p)=\mathrm{Vect}(e_1,e_2)$ and concludes the proof of claims 3 and 4 in the same way than before. $\square$

A point $x\in M$ is said *Oseledec-regular* if there exists $k\in\mathbb{N}^*$, real numbers $\lambda_1>\cdots>\lambda_k$ and a splitting $\mathrm{T}_xM=\oplus_{i=1}^kE_i$ such that for any $1\leq i\leq k$ and $u\in E_i\setminus\{0\}$:

$$\lambda_-(x,u)=\lambda_+(x,u)=\lambda_i.$$

The theorem of Oseledec draws a deep link between regular points and the ergodic theory of (φ^t), stating that the set $\mathcal{R}$ of Oseledec-regular points is a Borelian and that for any (φ^t)-invariant Borel probability measure μ on M: $\mu(\mathcal{R})=1$ (see for instance [Via14, Theorem 4.2] or [Can19, Théorème 4.1]). The *support* $\mathrm{supp}(\mu)$ of μ is defined as the set of points $x\in M$ such that for any neighbourhood U of x, $\mu(U)>0$.

Corollary 4.11. *1. The set $\mathcal{R}$ of Oseledec-regular points of (φ^t) is contained in the closed set $F=\mathcal{T}^-\cup\Upsilon^-\cup\mathcal{T}^+\cup\Upsilon^+$ of empty interior.*
2. The support of any (φ^t)-invariant Borel proability measure on M is contained in F and has thus empty interior.

Proof. 1. Let x be an Oseledec-regular point with Lyapunov splitting $\mathrm{T}_xM=\oplus_{i=1}^kE_i$ and associated Lyapunov exponents $\lambda_1>\cdots>\lambda_k$. Then denoting $V_i^+=\oplus_{j=i}^kE_j$ for $1\leq i\leq k$ (and $V_{k+1}^+=\{0\}$), for any $1\leq i\leq k$ and $u\in V_i^+\setminus V_{i+1}^+$: $\lambda_+(x,u)=\lambda_i$. Likewise denoting $V_i^-=\oplus_{j=1}^iE_j$ for $1\leq i\leq k$ (and $V_0^-=\{0\}$), for any $1\leq i\leq k$ and $u\in V_i^-\setminus V_{i-1}^-$: $\lambda_-(x,u)=\lambda_i$ (see for instance [Mañ, Chapter 4 §10]). In particular $\lambda_\varepsilon(x,u)$ exists for any $u\in\mathrm{T}_xM\setminus\{0\}$ and $\varepsilon=\pm1$, and

$$\{\lambda_+(x,u)\mid u\in\mathrm{T}_xM\setminus\{0\}\}=\{\lambda_-(x,u)\mid u\in\mathrm{T}_xM\setminus\{0\}\}$$

since both sets are equal to $\{\lambda_1,\ldots,\lambda_k\}$. The Proposition 4.10 shows that these sets are different at any point x outside of $\mathcal{T}^-\cup\Upsilon^-\cup\mathcal{T}^+\cup\Upsilon^+$, and thus that $\mathcal{R}$ is contained in $\mathcal{T}^-\cup\Upsilon^-\cup\mathcal{T}^+\cup\Upsilon^+$ (that was shown in Proposition 4.9 to be closed and of empty interior).

2. For any such measure μ, $\mu(\mathcal{R}) = 1$ according to Oseledec's theorem, and thus $\mu(\mathcal{T}^- \cup \Upsilon^- \cup \mathcal{T}^+ \cup \Upsilon^+) = 1$ according to the first part of the Corollary, which implies that $\mathrm{supp}(\mu) \subset \mathcal{T}^- \cup \Upsilon^- \cup \mathcal{T}^+ \cup \Upsilon^+$ since the latter set is closed (see for instance [Cou16, Proposition 18.2]). $\square$

This concludes the proof of Theorem A.

4.4.4. *New essential automorphisms of path structures.* In particular, we deduce from the previous results the following properties of the flow (φ^t).

Proposition 4.12. *For any $t \neq 0$:*

1. *the group generated by φ^t is not relatively compact for the compact-open topology;*
2. *φ^t is not a partially hyperbolic diffeomorphism of M.*

Proof. 1. This is a direct consequence of Proposition 4.9.
2. Up to conjugation of Γ in $j(\mathrm{SL}_2(\mathbb{R}))$ we can assume that $x_0 \coloneqq ((e_1,),(e_1,e_2)) \in \hat{\Omega}$. Then $f(x_0) = x_0$, and we saw in the proof of Proposition 4.10 that $\mathrm{D}_{x_0}\varphi^t$ is conjugated in the chart ϕ_1 (see (4.3)) to the diagonal matrix $\mathrm{Diag}(1, \mathrm{e}^{-t}, \mathrm{e}^{-t})$. This matrix being not partially hyperbolic, this shows that f is not partially hyperbolic. $\square$

We recall that an automorphism flow (φ^t) of a path structure (E^α, E^β) is said to be *strongly essential* if it does not preserve any continuous one-dimensional distribution transverse to $E^\alpha \oplus E^\beta$.

Proposition 4.13. *(φ^t) is a strongly essential flow of the path structure $(M, \mathcal{L})$.*

Proof. We assume by contradiction that a continuous transverse distribution does exist, and we consider its pullback on $\hat{\Omega}$ by Π. This is a (ψ^t)-invariant continuous one-dimensional distribution on $\hat{\Omega}$ denoted by E^c, transverse to the contact distribution of $\mathcal{L}_{\hat{\mathbf{X}}} = (E^\alpha, E^\beta)$. We saw in Paragraph 4.2 that $\Omega = \mathbf{X} \setminus \cup_{\gamma_\infty \in \partial_\infty \Gamma}(\mathcal{C}_\alpha(p_+(\gamma_\infty)) \cup \mathcal{C}_\beta[p_+(\gamma_\infty), e_3])$. In particular, up to conjugation of Γ in $j(\mathrm{GL}_2(\mathbb{R}))$ we can assume in this proof that both $x_0 \coloneqq (e_1,(e_1,e_2))$ and $y_0 \coloneqq (e_3,(e_3,e_2))$ are points of $\hat{\Omega}$. We consider the chart $\phi_1\colon \mathbb{R}^3 \to \hat{\mathbf{X}}$ defined in (4.3) around $x_0 = \phi_1(0,0,0)$. The closed subset $K = \{x \in \mathbb{R} \mid [1,x,0] \in \cup_{\gamma_\infty \in \partial_\infty \Gamma} p_+(\gamma_\infty)\}$ verifies $0 \notin K$ (since $x_0 \in \hat{\Omega}$) and $\phi_1^{-1}(\hat{\Omega}) = \mathbb{R}^3 \setminus (K \times \{0\} \times \mathbb{R})$. We have $\phi_1^*(E^\alpha \oplus E^\beta) = \mathrm{Ker}(zdx - dy)$, and we denote $E_1^c = \phi_1^* E^c$. For any $(\lambda,\mu) \in (\mathbb{R}^*)^2$:

$$\phi_1^{-1} \circ \mathrm{Diag}(\lambda,\mu,1) \circ \phi_1 = \mathrm{Diag}(\lambda^{-1}\mu, \lambda^{-1}, \mu^{-1}). \tag{4.5}$$

We can conjugate Γ in the stabilizer of (x_0,y_0) in $j(\mathrm{GL}_2(\mathbb{R}))$, equal to $\{\mathrm{Diag}(\lambda,\mu,1)\}$. But (4.5) shows that this stabilizers acts transitively on the tangent directions D at $(0,0,0)$ transverse to $\mathrm{Vect}(e_1,e_3) = \phi_1^*(E^\alpha \oplus E^\beta)(0,0,0)$ and distinct from $\mathbb{R}e_2$. Up to conjugation in $j(\mathrm{GL}_2(\mathbb{R}))$, we can thus assume that $E_1^c(0,0,0) \neq \mathbb{R}(0,1,1)$, which will be important later.

We denote $A^t \coloneqq \phi_1^{-1} \circ \psi^t \circ \phi_1 = \mathrm{Diag}(1, \mathrm{e}^{-t}, \mathrm{e}^{-t})$. For any $(x,y,z) \in O \coloneqq \{(x,y,z) \in \mathbb{R}^3 \mid x \notin K\}$, denoting $E_1^c(x,y,z) = \mathbb{R}(a,b,c)$ we have $E_1^c(x, \mathrm{e}^{-t}y, \mathrm{e}^{-t}z) = \mathrm{D}_{(x,y,z)}A^t(E_1^c) = \mathbb{R}(a, \mathrm{e}^{-t}b, \mathrm{e}^{-t}c)$ by A^t-invariance (note that O is (A^t)-invariant). If $a \neq 0$, then $E_1^c(x,0,0) = \mathbb{R}e_1$ by continuity of E_1^c, which contradicts the transversality with $\phi_1^*(E^\alpha \oplus E^\beta)$. Hence $a = 0$, $b \neq 0$ by transversality with $\phi_1^*(E^\alpha \oplus E^\beta)$, $E_1^c(x, \mathrm{e}^{-t}y, \mathrm{e}^{-t}z)$ is equal to $\mathbb{R}(0,1,\frac{c}{b})$ for any $t \in \mathbb{R}$ and $E_1^c(x,0,0) = \mathbb{R}(0,1,\frac{c}{b})$ by continuity of E_1^c. There exists thus a continuous $\mathbb{R}$-valued function λ on $\mathbb{R} \setminus K$ such that $E_1^c(x,y,z) = \mathbb{R}(0,1,\lambda(x))$ for any $(x,y,z) \in O$. Furthermore, $\lambda(0) \neq 1$ since $E_1^c(0,0,0) \neq \mathbb{R}(0,1,1)$.

We now consider the chart $\phi_2\colon \mathbb{R}^3 \to \hat{\mathbf{X}}$ around $y_0 = \phi_2(0,0,0)$ defined in (4.4). With $U \coloneqq \phi_2^{-1}(\phi_1(\mathbb{R}^3)) = \{x \neq 0, zyx^{-1} \neq 1\}$ and $V \coloneqq \phi_1^{-1}(\phi_2(\mathbb{R}^3)) = \{y \neq 0, zxy^{-1} \neq 1\}$, the transition maps $\phi_2^{-1} \circ \phi_1\colon V \to U$ and $\phi_1^{-1} \circ \phi_2\colon U \to V$ are given by

$$\phi_2^{-1} \circ \phi_1(x,y,z) = \left(y^{-1}, xy^{-1}, \frac{zy^{-1}}{zxy^{-1}-1}\right), \phi_1^{-1} \circ \phi_2(x,y,z) = \left(yx^{-1}, x^{-1}, \frac{zx^{-1}}{zyx^{-1}-1}\right).$$

Denoting $E_2^c = \phi_2^* E^c$, since $E_1^c(x,y,z) = \mathbb{R}(0,1,\lambda(x))$ a straightforward calculation gives

$$E_2^c(x,y,z) = \mathbb{R}(x^2, xy, x(z - \lambda(yx^{-1}))(zyx^{-1}-1)^2)$$

for any $(x, y, z) \in \phi_2^{-1} \circ \phi_1(O)$. Now for $0 < x < 1$ small enough, $(x, x^2, 1) \in \phi_2^{-1} \circ \phi_1(O)$, and $E_2^c(x, x^2, 1) = \mathbb{R}(x^2, x^3, x(1-\lambda(x))(x-1)^2) = \mathbb{R}(x, x^2, (1-\lambda(x))(x-1)^2)$ converges at $x = 0$ to $\mathbb{R}(0, 0, (1-\lambda(0))) = \mathbb{R}e_3$ since $\lambda(0) \neq 1$. Hence $E_2^c(0, 0, 1) = \mathbb{R}e_3$ by continuity, but $\mathbb{R}e_3 = (\phi_2^* E^\alpha)(0, 0, 1)$. This contradicts the transversality of E^c and $E^\alpha \oplus E^\beta$ and concludes the proof. $\square$

4.5. **About other compactifications of** $\mathrm{T}^1\Sigma$**.** We conclude this paper by describing two other geometrical compactifications of $\mathrm{T}^1\Sigma$.

4.5.1. *A second path structure on* $\mathrm{T}^1\Sigma$. We defined in Paragraph 1.1.2 of the introduction a natural path structure $\mathcal{L}_S^{proj}$ defined on the unitary tangent bundle of any Riemannian surface S. The *projective class* of the metric of S, that is its set of unparametrized geodesics, is actually sufficient to define an analog path structure on the projectivization $\mathbf{P}(\mathrm{T}S)$ of the tangent bundle of S (that we will still denote $\mathcal{L}_S^{proj}$ by a slight misuse of notations). The natural two-sheeted covering $\mathrm{T}^1 S \to \mathbf{P}(\mathrm{T}S)$ is a local isomorphism between these path structures.

If Σ is a non-compact hyperbolic surface, finding a compactification of $(\mathbf{P}(\mathrm{T}\Sigma), \mathcal{L}_\Sigma^{proj})$ is thus equivalent to find a *projective compactification* of Σ, that is a compact projective surface S with a *projective copy* of Σ – an open subset $U \subset S$ and a diffeomorphism between U and Σ mapping unparametrized geodesics to unparametrized geodesics. Such a projective compactification is given by [CG17, Theorem 1.1]. It is interesting to note that the projective compactification S constructed by Choi-Goldman contains two disjoint projective copies of the surface Σ and that $(\mathbf{P}(\mathrm{T}S), \mathcal{L}_S^{proj})$ contains thus two disjoint copies of $(\mathbf{P}(\mathrm{T}\Sigma), \mathcal{L}_\Sigma^{proj})$, which is reminiscent of the four copies of $(\mathrm{T}^1\Sigma, \mathcal{L}_\Sigma)$ appearing in Theorem A and raises the following:

Question c. Does there exist a projective compactification containing a dense copy of the complete non-compact hyperbolic surface Σ ?

4.5.2. *A conformal Lorentzian compactification of* $\mathrm{T}^1\Sigma$. We recall from Paragraph 1.1.2 that unlike the path structure $\mathcal{L}_\Sigma = (E^s, E^u)$ that we have studied in the whole section 4, the previous path structure $\mathcal{L}_\Sigma^{proj}$ is *not* invariant by the geodesic flow (g^t) of a complete hyperbolic surface Σ. Now consider the one-form θ defined on $\mathrm{T}^1\Sigma$ by $\theta|_{E^s \oplus E^u} \equiv 0$ and $\theta(X^c) \equiv 1$, where $X^c = \frac{dg^t}{dt}$. Then $h_\Sigma(v^s, v^u) = h_\Sigma(v^u, v^s) := d\theta(v^s, v^u)$ for any $(v^s, v^u) \in E^s \times E^u$ and $h_\Sigma(E^s \oplus E^u, X^c) = 0$ defines on $\mathrm{T}^1\Sigma$ a *Lorentzian metric* h_Σ. This third geometric structure on $\mathrm{T}^1\Sigma$ is actually closer to the focus of the present work, as the geodesic flow (g^t) acts by isometries of h_Σ. One can weaken this structure by considering its *conformal class* $[h_\Sigma]$, that is the set of all Lorentzian metrics $\mathrm{e}^f h_\Sigma$ for $f\colon \mathrm{T}^1\Sigma \to \mathbb{R}$ a smooth function. If Σ is non-compact, Frances describes in [Fra05, §4.6] (as a consequence of a more general work about conformal Lorentzian structures) a *conformal compactification* of $(\mathrm{T}^1\Sigma, [h_\Sigma], g^t)$, where the geodesic flow extends to a flow of conformal automorphisms (diffeomorphisms preserving the conformal class). We emphasize that unlike the compactifications for the path structures $\mathcal{L}_\Sigma$ and $\mathcal{L}_\Sigma^{proj}$, the image of $\mathrm{T}^1\Sigma$ is a dense subset of its Lorentzian conformal compactification.

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Martin Mion-Mouton, Department of Mathematics, Technion, Haifa 32000, Israel.
Email address: `martinm@campus.technion.ac.il`
URL: `https://martinm.webgr.technion.ac.il/`

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