My research works and projects lie at the intersection between geometric structures, dynamical systems and group actions. On the one hand, I am interested in differentiable dynamical systems verifying rigid geometric assumptions. I obtained a classification result for three-dimensional partially hyperbolic diffeomorphisms of contact type, by studying an associated invariant rigid geometric structure named Lagrangian contact structure, that naturally fall within the framework of Cartan geometries. This led me, on the other hand, to study the action of “Schottky” discrete subgroups of $\text{PGL}_3(\mathbb{R})$ on the three-dimensional flag space. By this way, I constructed a compactification of a family of Lagrangian contact structures, as well as new examples of automorphisms of these structures. In this research project, I introduce first the principal aspects of my past work, before describing my future projects, concerning each of the both directions of research depicted in this introduction.

1. State of the art and contributions

1.1. Contact-Anosov flows. Let us recall that a non-singular flow $(\varphi^t)$ of class $C^\infty$ of a compact manifold $M$ is called Anosov, if its differential preserves a splitting $TM = E^s \oplus E^0 \oplus E^u$ of the tangent bundle, where $E^0$ is the direction of the flow, and where $E^s$ and $E^u$ are non-trivial distributions verifying the following estimates (with respect to any Riemannian metric on $M$).

1. The stable distribution $E^s$ is uniformly contracted by $(\varphi^t)$, i.e. there are two constants $C > 0$ and $0 < \lambda < 1$ such that for any $t \in \mathbb{R}$ and $x \in M$:

$$\left\| D_x \varphi^t \right\|_{E^s} \leq C \lambda^t.$$  

2. The unstable distribution $E^u$ is uniformly expanded by $(\varphi^t)$, i.e. uniformly contracted by $(\varphi^{-t})$.

Important examples of three-dimensional Anosov flows are given by the geodesic flows of compact hyperbolic surfaces $\Sigma$, acting on the unitary tangent bundle $T^1_0 \Sigma$. These flows have the following specific properties among Anosov flows: their stable and unstable distributions are $C^\infty$ (while they are in general only Hölder continuous), and the sum $E^s \oplus E^u$ is furthermore a contact distribution (i.e., roughly, is nowhere integrable). A beautiful result of Étienne Ghys in [Ghy87] says that, up to orbital equivalences and finite coverings, the geodesic flows of compact hyperbolic surfaces are in fact the only examples of three-dimensional Anosov flows whose stable and unstable distributions are $C^\infty$ and such that $E^s \oplus E^u$ is a contact distribution. Ghys is actually more specific, and proves that all these flows are smoothly conjugated to an algebraic example (the right diagonal flow on a compact quotient of $\text{PSL}_2(\mathbb{R})$). We will thus call them the algebraic contact-Anosov flows of dimension three in this text. In 1992, Ghys theorem was generalized by Benoist, Foulon and Labourie in [BFL92]: they prove that any contact-Anosov flow with smooth stable and unstable distributions is, up to finite coverings, $C^\infty$-orbitally equivalent to the geodesic flow of a compact locally symmetric Riemannian manifold of negative curvature.

1.2. Partially hyperbolic diffeomorphisms of contact type. The results of [Ghy87] and [BFL92] are striking expressions of the dynamical rigidity that can be deduced from geometrical assumptions, for the case of continuous-time dynamical systems. A thrilling question is then to know if these results generalize for discrete-time dynamical systems. Natural discrete-time analog for the Anosov flows are the diffeomorphisms $f$ of compact manifolds $M$, such that $Df$ preserves a splitting $TM = E^s \oplus E^c \oplus E^u$, such that $E^s$ (respectively $E^u$) is uniformly contracted (resp.
for various reasons. In this whole text, we will say that a partially hyperbolic diffeomorphism \( f \) is of contact type, if its invariant distributions \( E^s, E^u \) and \( E^c \) are smooth, and if \( E^s \oplus E^c \) is a contact distribution.

**Theorem A** ([MM20a, Theorem A]). Let \( f \) be a partially hyperbolic diffeomorphism of contact type of a three-dimensional connected compact manifold \( M \). If the non-wandering set \( NW(f) \) equals \( M \), then up to a finite quotient or covering of \( M \) and up to finite iterates, \( f \) is \( C^\infty \)-conjugated to one of the following examples:

1. the time-one map of a three-dimensional algebraic contact-Anosov flow;
2. or a partially hyperbolic affine automorphism of a \( \text{nil-Heis}(3) \)-manifold.

The second family of examples is given by partially hyperbolic diffeomorphisms induced on a compact quotient \( \Gamma \backslash \text{Heis}(3) \) by an affine automorphism of \( \text{Heis}(3) \) preserving a lattice \( \Gamma \) (see for instance [Sma67, Ham13] for a description of such algebraic examples).

Actually, Theorem A is obtained for more general diffeomorphisms than partially hyperbolic ones. First of all, Theorem A remains true while removing the hypothesis of domination on the splitting (no dynamical assumption is needed on \( E^c \), see [MM20a, Corollary 8.2]). Furthermore, the classification does not rely on any uniformity concerning the contraction (respectively expansion) of \( E^s \) (resp. \( E^u \)). More precisely, let \( f \) be a diffeomorphism of a three-dimensional connected compact manifold \( M \), preserving a smooth splitting \( TM = E^\alpha \oplus E^c \oplus E^\beta \), such that \( E^\alpha \oplus E^\beta \) is a contact distribution, \( f \) has a dense orbit on \( M \) (this replaces the hypothesis \( NW(f) = M \)), and for any \( x \in M \) we have, for \( \varepsilon = \alpha \) and \( \varepsilon = \beta \):

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(1.2) \quad \lim_{n \to +\infty} \|D_x f^n|_{E^\varepsilon}\| = 0 \quad \text{or} \quad \lim_{n \to +\infty} \|D_x f^n|_{E^\varepsilon}\| = 0
\]

with respect to some Riemannian metric on \( M \). Then the conclusions of Theorem A hold on \( f \) (see [MM20a, Theorem B]). Let us emphasize that the assumption (1.2) is closely related to the notion of quasi-Anosov diffeomorphism of Mañé in [Mañ77].

### 1.3. Lagrangian contact structures and Cartan geometries.

The triplet \( S = (E^s, E^c, E^u) \) preserved by a partially hyperbolic diffeomorphism \( f \) of contact type happens to be a rigid geometric structure, and the rough idea is then the following: the geometrical classification of \( S \) will be sufficient to obtain the dynamical classification of \( f \), since \( f \) is an automorphism of \( S \).

On a three-dimensional manifold, a couple \( \mathcal{L} = (E^\alpha, E^\beta) \) of one-dimensional smooth distributions whose sum is a contact distribution, is called a Lagrangian contact structure. These structures are intimately linked with the homogeneous space \( X \) of full flags of \( \mathbb{R}^3 \), endowed with a natural Lagrangian contact structure invariant under the action of \( \text{PGL}_3(\mathbb{R}) \) on \( X \), that plays for them the same role that the euclidean space plays for Riemannian metrics. The notion of Cartan geometry (originally due to Élie Cartan, see [Car10, Sha97, ČS09]) allows indeed to give a precise meaning to the following idea: every three-dimensional Lagrangian contact structure \( \mathcal{L} \) is a “curved version” of the homogeneous space \( X \), and enjoys a curvature whose vanishing is equivalent for \( \mathcal{L} \) to be flat (i.e. locally isomorphic to the model space \( X \)).

In the case of partially hyperbolic diffeomorphisms of contact type, the Cartan geometry associated to the Lagrangian contact structure \( \mathcal{L} = (E^s, E^u) \) plays a crucial role in the comprehension of the \( f \)-invariant geometric structure \( S = (E^s, E^c, E^u) \).

### 1.4. Compactifications of Lagrangian contact structures.

As rigid geometric structures, three-dimensional Lagrangian contact structures defined on a compact manifold and whose automorphism group is non-compact (for the compact-open topology) are expected to be particularly peculiar: possibly locally homogeneous, or maybe even flat.\(^3\) From a generic point of view, Barbot\(^2\)An important one being a better understanding of robust dynamical behaviours.

\(^3\)This is a particular case, for Lagrangian contact structures, of a remark concerning all rigid geometric structures. See [LF69, Oba71] for a paradigmatic result for conformal Riemannian structures, and [GD91] for a general point of view.
constructed in [Bar10] a large family of $(\text{PGL}_3(\mathbb{R}), \mathbf{X})$-structures, whose holonomies are Anosov representations of a surface group into $\text{PGL}_3(\mathbb{R})$. But by nature, this method does not allow to control the automorphism groups of the resulting structures.

All the examples of Theorem A being conservative (i.e. preserving a smooth volume form), a first reasonable problem to understand the diversity of Lagrangian contact structures with large automorphism groups, is to find non-conservative examples of Lagrangian contact automorphisms that are non-equi-continuous (i.e. that generate a non-compact group). For any (complete) hyperbolic surface $\Sigma$, the unitary tangent bundle $T^1\Sigma$ is endowed with a natural Lagrangian contact structure $\mathcal{L}_\Sigma$ invariant by the geodesic flow, for which I obtained the following result yielding new examples of Lagrangian contact structures with non-compact automorphism groups.

**Theorem B** ([MM20b, Theorem B p.8]). Let $\Sigma$ be a non-compact hyperbolic surface whose fundamental group is finitely generated. Then, there exists a (possibly different) hyperbolic metric on the underlying topological surface, for which we have the following.

1. The Lagrangian-contact structure $(T^1\Sigma, \mathcal{L}_\Sigma)$ admits a compactification $(M, \mathcal{L})$.

2. Furthermore, the geodesic flow of $T^1\Sigma$ extends to a non-equi-continuous and non-conservative automorphism flow of $(M, \mathcal{L})$.

## 2. Ongoing and Future Projects

### 2.1. Rigidity of three-dimensional partially hyperbolic diffeomorphisms.

Ghys actually classifies in [Ghy87] all three-dimensional Anosov flows with smooth stable and unstable distributions. A natural project is then to extend Theorem A, by classifying all the three-dimensional partially hyperbolic diffeomorphisms having smooth invariant distributions $E^s$, $E^c$ and $E^u$. Recently, a result was obtained in this direction in [CPRH19], but the authors make the following strong additional restriction on the partially hyperbolic diffeomorphism $f$. In any global frame of vector fields generating $(E^s, E^c, E^u)$, $D_x f$ reads as a diagonal matrix, and they assume that there exists a global frame where the coefficients of this matrix, called the exponents of $f$, are constant.

In Theorem A, the following two additional assumptions are made, compared with the general classification problem of three-dimensional partially hyperbolic diffeomorphisms with smooth invariant distributions: $NW(f) = M$, and $E^s \oplus E^u$ contact. Beyond the interest of a general result, the elimination of these two additional assumptions raise two different interesting geometrical and dynamical problems, that we successively describe in the next two paragraphs.

#### 2.1.1. Beyond the conservative setting.

The two family of diffeomorphisms appearing in Theorem A are conservative (and have thus no wandering points). To the best of my knowledge, every known example of partially hyperbolic diffeomorphism of contact type is actually conservative (and is therefore one of the examples of Theorem A). According to an argument of Brin in [Bri75], for a partially hyperbolic diffeomorphism $f$ of contact type, if $NW(f)$ has a non-empty interior then $f$ is topologically transitive. It has thus no wandering points, and it is therefore conservative according to Theorem A. For partially hyperbolic diffeomorphisms of contact type, $\text{Int}(NW(f)) \neq \emptyset$, $NW(f) = M$, topological transitivity and $f$ conservative are finally equivalent assumptions, and most of classification results for partially hyperbolic diffeomorphisms are proved under one of these assumptions.\(^4\) For this reason, it would be interesting to answer the following question (unsolved yet): is every partially hyperbolic diffeomorphism of contact type conservative?\(^5\)

According to [CPRH19], one strategy could be to prove that the exponents (in the sense defined above) of such a diffeomorphism are necessarily constant on its non-wandering set (without hypothesis on its interior). The Cartan geometry of $(E^s, E^u)$ could allow to provide a positive answer for the central exponent. On a geometrical point of view, the background of this problem is the study of a rigid geometric structure (here, $\mathcal{L} = (E^s, E^u)$) for which the relevant information

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\(^4\)For instance in [CPRH19], it is assumed topological transitivity or real analyticity of $f$, in the case of a non-Anosov diffeomorphism.

\(^5\)Let us emphasize that this is automatically the case for contact-Anosov flows.
is on a closed subset of possible empty interior (here, $NW(f)$), which is a general interesting problem.

2.1.2. The non-contact case. A second natural question is to relax the geometrical hypothesis, by still assuming the invariant distributions $(E^s, E^c, E^u)$ of the three-dimensional partially hyperbolic diffeomorphism $f$ to be smooth, but only assuming the open subset $O \subset M$ where $E^s \oplus E^u$ is contact to be non-empty (the case $O = M$ being the one of contact-type partially hyperbolic diffeomorphisms). The only known examples for which $O \subseteq M$ are time-one maps of Anosov flows (as it appears in [CPRH19]), and according to [Ghy87] they are therefore, up to finite coverings and iterates, $C^{\infty}$-conjugated to the suspension of an Anosov automorphism of the two-torus. In this case, the distribution $E^s \oplus E^u$ is integrable and therefore $O$ is empty, which suggest the following behaviour: $O \neq \emptyset$ implies $O = M$. A possible strategy to prove this conjecture would be to show on the one hand that the Lagrangian contact structure $\mathcal{L} = (E^s, E^u)$ (defined on $O$) necessarily extends to $M$, and to prove on the other hand that $\mathcal{L}$ has to be maximal for dynamical reasons.

This is a particular case of a more general interesting geometric problem, that of a “degenerated” geometric structure. The Lagrangian contact structure $\mathcal{L} = (E^s, E^u)$ is degenerating on the boundary $\partial O$, and even if it is not possible to directly obtain $O = M$, I wonder if it is possible to use the geometrical informations on $O$ to constrain $f$ on the whole manifold. More precisely, are the exponents of $f$ necessarily constant on $O$? A positive answer would provide a first step for the general classification of three-dimensional partially hyperbolic diffeomorphisms with smooth distributions. A possible strategy could be to consider another $f$-invariant geometric structure in parallel to $\mathcal{L}$. A candidate is for instance a generalized conformal structures as studied in [BZ17].

2.2. Higher-dimensional situation. I also want to address the corresponding problem in the higher-dimensional situation, i.e. to look for an analog to Benoist-Foulon-Labourie Theorem for partially hyperbolic diffeomorphisms $f$ of contact type in any (odd) dimension. In this situation, the couple $\mathcal{L} = (E^s, E^u)$ remains a $f$-invariant (higher-dimensional) Lagrangian contact structure, and the tools offered by the normal Cartan geometry of $\mathcal{L}$ will undoubtedly be of great help to understand $f$. However, the whole classification seems far to be achieved for the moment.

First of all, this would require to understand precisely the possible candidates for the higher-dimensional partially hyperbolic diffeomorphisms of contact type, as well as the geometry of their associated Lagrangian contact structures $(E^s, E^u)$. Furthermore, the current scheme of proof of Theorem A heavily relies on a result specific to the dimension three, in which any locally homogeneous Lagrangian contact structure with non-trivial isotropy is flat (see [Tre96] or [MM20a, Theorem 3.3]). In contrast, the structure $(E^s, E^u)$ is non-flat for some higher-dimensional examples, which already suggest the diversity of the possible geometries.

A fundamental result of Gromov known as the “open-dense orbit Theorem” claims that in the presence of an automorphism having a dense orbit, a rigid geometric structure is locally homogeneous in restriction to an open and dense subset. Since a partially hyperbolic diffeomorphism $f$ of contact type with no wandering points is topologically transitive according to [Bri75], the $f$-invariant geometric structure $S = (E^s, E^c, E^u)$ is thus locally homogeneous on an open-dense subset $\Omega \subset M$. If $\Omega$ were equal to $M$, then the problem would be reduced to the following two-part strategy: understanding the possible local models and then dealing with the global structure via the developing map of the resulting $(G, X)$-structures. A first important task is thus to understand if $\Omega = M$, which is a particular case of the following classical question for rigid geometric structures: can the maximal open subset of local homogeneity given by Gromov be a strict subset of $M$?\footnote{See [Gro88], and [Pec16] for the specific case of Cartan geometries.}

2.3. Ping-pong dynamics and compactifications. A third aspect of my future research projects concerns the understanding of Lagrangian contact structures with non-compact automorphism groups. This is a two-sided problem: try to construct new examples on the first hand,
and try to prove general properties to restrict the possibilities on the other hand. Currently, I consider that a lot of new examples are still to be discovered and I am thus focused on the first aspect.

Theorem B is obtained through the description of the dynamic, on the flag space $\mathbf{X}$, of discrete subgroups of $\text{PGL}(\mathbb{R})$ that are the holonomy groups of the flat Lagrangian contact structure $\mathcal{L}_\Sigma$ (see paragraph 1.4). These subgroups have some kind of ping-pong dynamics on $\mathbf{X}$, whose precise description of the attracting and repelling objects in $\mathbf{X}$ allows to construct a compactification. I am currently trying to obtain general criteria for a (flat) Lagrangian contact structure either to admit a compactification, or conversely to be maximal (which is a natural and important question to ask for any geometric structure, see for instance [Fra12] in the general setting of Cartan geometries). Such criteria could have dynamical implications, by showing that the “contact locus” introduced in paragraph 2.1.2, where the distribution $E_s^u \oplus E^u$ induced by a partially hyperbolic diffeomorphism is contact, equal the whole manifold. This work has also connections with other geometric structures.

I have indeed the project to use the method of compactification developed for Theorem B, to construct a $(\text{PGL}(\mathbb{R}), \mathbf{X})$-structure on a compact hyperbolic three-dimensional manifold (the existence of such a structure seems to be an open problem so far). In [FT15], Falbel and Thebaldi develop a general method of construction of such structures (via gluings of tetrahedra in $\mathbf{X}$), which can be seen as an analog for Lagrangian contact structures of Thurston’s construction of hyperbolic structures on the complement of a knot in $\mathbb{S}^3$ (see [Thu97]). They obtain by this way a $(\text{PGL}(\mathbb{R}), \mathbf{X})$-structure on an open complete hyperbolic manifold, and Elisha Falbel kindly suggested to me that the method used in Theorem B could be used to “compactify” this example.

References


[10] Which is intimately connected with the notion of Anosov representations, see for instance [Bar10, GGKW17].


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