PATH-LIFTING AND COVERINGS

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ABSTRACT. In this short text, we first recall a classical result which characterizes coverings by the path-lifting property, and we then prove that in presence of a splitting within one-dimensional distributions, it suffices to lift the paths tangent to these directions. We emphasize that this second statement is probably as classical as the first one, and certainly not new.

1. Characterization of coverings by path-lifting

In this text, a *path* is a continuous map from a closed interval of $\mathbb{R}$ to a metric space. If $f\colon M \to N$ is a continuous map between two metric spaces, we say that a path $\gamma\colon [a\,;b] \to N$ *starting from* $x = \gamma(a) \in N$ *lifts* to M *from* $\tilde{x} \in f^{-1}(x)$ (*through* f), if there exists a path $\tilde{\gamma}\colon [a\,;b] \to N$ (the *lift* of γ) such that $f \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(a) = \tilde{x}$. Note that if f is a local homeomorphism from M to N, then two lifts of a same path that coincide at a point are equal.

The following result is classical, and proved in different algebraic topology books. For convenience, we give here a proof.

Lemma 1. *Let $f\colon M \to N$ be a local homeomorphism between two topological manifolds, N being connected. Then f is a covering map if, and only if for any $x \in \mathrm{Im}(f)$ and $\tilde{x} \in f^{-1}(x)$, any path starting from x lifts through f from $\tilde{x}$.*

Remarks 2.
1. N being arcwise connected, if f verifies the second assumption called the *path lifting property*, then f is automatically surjective.
2. It seems to us that the proof does actually not use the fact that M and N are topological manifolds, but only that: the metric spaces M and N are locally arcwise connected, and any point of N admits a connected and simply connected neighbourhood.

Proof of Lemma 1. The only if part is well known. We now assume that f verifies the path lifting property, and denote $n = \dim M = \dim N$. Let $x \in N$ and U be an open neighbourhood of x which is connected and simply connected (for instance, choose U to be homeomorphic to the open ball $B^n \subset \mathbb{R}^n$). The restriction of f from $f^{-1}(U)$ to U still verifies the hypotheses of the Lemma, and from now on we replace M by $f^{-1}(U)$, N by U and f by $f|_U$, so that f is a surjective local homeomorphism from a manifold M to a simply connected manifold N, verifying the path-lifting property. Denoting by M_i the arcwise connected components of M, we only need to show that for any i, $f|_{M_i}\colon M_i \to N$ is a homeomorphism to conclude the proof. Note that, M_i being connected, if $x \in f(M_i)$, $\tilde{x} \in f^{-1}(x)$, γ is a path in N starting from x and $\tilde{\gamma}$ its lift from $\tilde{x}$, then $\tilde{\gamma}$ is contained in M_i. In other words, $f|_{M_i}\colon M_i \to N$ still verifies the path lifting property, and in particular $f(M_i) = N$. Finally, we can assume M to be arcwise connected, and we only need to show that f is injective.

Let x, x' in M such that $f(x) = f(x') = y$, and let $\alpha\colon [0\,;1] \to M$ be a continuous path from $x = \alpha(0)$ to $x' = \alpha(1)$. Then $\bar{\alpha} = f \circ \alpha$ is a continuous loop based at y, and N being simply connected there exists a continuous homotopy $H\colon [0\,;1]^2 \to N$ from $\bar{\alpha} = H(0,\cdot)$ to the constant path at y denoted by $\bar{\beta} = H(1,\cdot)$, verifying $H(s,0) = H(s,1) = y$ for any $s \in [0\,;1]$.

Fact 3. *There exists a continuous map $G\colon [0\,;1]^2 \to M$ such that $f \circ G = H$ and $G(0,0) = x$.*

Proof. For any $s \in [0\,;1]$, there exists a continuous lift $\tilde{H}_s\colon [0\,;1] \to M$ of $H(s,\cdot)$ from x, and we define $G(s,t) = \tilde{H}_s(t)$. This map verifies $f \circ G = H$ and $G(s,0) = x$ for any $s \in [0\,;1]$. We are going to prove that G is continuous, by proving that $E = \{t \in [0\,;1]; \forall s \in [0\,;1], G \text{ is continuous on a neighbourhood of } (s,t)\}$ is open, closed and non-empty. The crucial property that we are going to use is of course that $G(s,\cdot)$ is continuous for any $s \in [0\,;1]$.

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If $t \in E$, then for any $s \in [0\,;1]$ there exists $\varepsilon_s > 0$ such that G is continuous on $([s-\varepsilon_s\,; s+\varepsilon_s] \cap [0\,;1]) \times [t-\varepsilon_s\,; t+\varepsilon_s]$. Since the $]s-\varepsilon_s\,; s+\varepsilon_s[\cap [0\,;1]$ cover the compact $[0\,;1]$, there exists a finite number of s_i such that $[0\,;1] \subset \cup_i]s_i - \varepsilon_{s_i}\,; s_i + \varepsilon_{s_i}[$. Taking for $\varepsilon > 0$ the minimum of these ε_{s_i}'s, we conclude that for any $s \in [0\,;1]$, G is continuous on an open neighbourhood of $\{s\} \times [t-\varepsilon\,; t+\varepsilon]$. In particular, $[t-\varepsilon\,; t+\varepsilon] \subset E$, which is thus open.

Let U be an open neighbourhood of x such that $f|_U$ is an homeomorphism from U to $V = f(U)$, neighbourhood of $y = f(x)$, and let $s \in [0\,;1]$. By continuity of H, since $H(s,0) = y$ there exists $\varepsilon > 0$ such that $H(([s-\varepsilon\,; s+\varepsilon] \cap [0\,;1]) \times [0\,;\varepsilon]) \subset V$. For any $s' \in [s-\varepsilon\,; s+\varepsilon] \cap [0\,;1]$, the paths $t \in [0\,;\varepsilon] \mapsto f|_U^{-1} \circ H(s',t)$ and $t \in [0\,;\varepsilon] \mapsto G(s',t)$ being two continuous lifts of $t \mapsto H(s,t)$ from x, they are equal. Therefore G coincide with $f|_U^{-1} \circ H$ on $([s-\varepsilon\,; s+\varepsilon] \cap [0\,;1]) \times [0\,;\varepsilon]$, and is in particular continuous on a neighbourhood of $(s,0)$, *i.e.* $0 \in E$.

Now we assume that (t_n) is an increasing sequence of points of E converging to $t > 0$, and we prove that $t \in E$. We fix $s \in [0\,;1]$, and choose an open neighbourhood U of $G(s,t)$ such that $f|_U$ is an homeomorphism from U to $V = f(U)$, neighbourhood of $H(s,t)$. There exists $\varepsilon > 0$ such that $H(([s-\varepsilon\,; s+\varepsilon] \cap [0\,;1]) \times [t-\varepsilon\,; t+\varepsilon]) \subset V$, and there exists by hypothesis $t' \in E \cap]t-\varepsilon\,; t[$. The paths $v \in [0\,; t-t'] \mapsto f|_U^{-1} \circ H(s, t-v)$ and $v \in [0\,; t-t'] \mapsto G(s,t-v)$ being two continuous lifts of $v \mapsto H(s,t-v)$ from $G(s,t)$, they are equal, and in particular $G(s,t') = f|_U^{-1} \circ H(s,t')$. Since $t' \in E$, G is continuous on a neighbourhood of (s',t') for any $s' \in [0\,;1]$, so the path $u \in [0\,;\varepsilon] \mapsto G(s+u,t')$ is locally continuous, and hence continuous. This is a continuous lift of $u \mapsto H(s+u,t')$ from $G(s,t')$, as is $u \mapsto f|_U^{-1} \circ H(s+u,t')$, so $G(u,t') = f|_U^{-1} \circ H(u,t')$ for any $s \in [s\,; s+\varepsilon]$. Doing the same with $u \in [0\,; -\varepsilon]$, we have $G = f|_U^{-1} \circ H$ in restriction to $([s-\varepsilon\,; s+\varepsilon] \cap [0\,;1]) \times \{t'\}$. Finally, for any $u \in ([s-\varepsilon\,; s+\varepsilon] \cap [0\,;1])$, $v \mapsto [0\,; t+\varepsilon-t'] \cap [0\,;1] \mapsto G(u,t'+v)$ and $v \mapsto f|_U^{-1} \circ H(u,t'+v)$ are two continuous lifts of $v \mapsto H(u,t'+v)$ from $G(u,t')$, and are thus equal. Doing the same with $[0\,; t'-(t-\varepsilon)]$, we finally conclude that G coincide with $f|_U^{-1} \circ H$ and is thus continuous on on $([s-\varepsilon\,; s+\varepsilon] \cap [0\,;1]) \times [t-\varepsilon\,; t+\varepsilon]$, showing that $t \in E$. This concludes the proof of this fact. ☐

Since $G(0,\cdot)$ and α are two continuous lifts from x of $\bar{\alpha}$, they are equal and $x' = \alpha(1) = G(0,1)$. Likewise, $G(1,\cdot)$ and the path constant equal to x being continuous two lifts from x of $\bar{\beta}$, they are equal and $x = G(1,1)$. But $G(\cdot,1)$ and the path constant equal to x' are two continuous lifts from x' of $H(\cdot,1)$ constant equal to y, and are thus equal, *i.e.* $x' = G(0,1) = G(1,1) = x$. This concludes the proof. ☐

2. Path-lifting in the directions of a splitting is sufficient

For diffeomorphisms, and in presence of a splitting of the tangent bundle of the basis within one-dimensional distributions, it is actually sufficient to lift the paths tangent to the directions of the splitting.

Lemma 4. *Let $f\colon M \to N$ be a local diffeomorphism between two smooth n-dimensional manifolds, N being connected. We assume that there is a smooth splitting $E_1 \oplus \cdots \oplus E_n = \mathrm{T}N$ of the tangent bundle of N into one-dimensional smooth distributions, and that for any $i \in \{1,\ldots,n\}$, $x \in \mathrm{Im}(f)$ and $\tilde{x} \in f^{-1}(x)$, any smooth path tangent to E_i and starting from x lifts through f to a path starting from $\tilde{x}$. Then f verifies the path lifting property, and is thus a covering map from M to N. In particular, f is* a posteriori *surjective.*

Note that if f is a local diffeomorphism, then any continuous lift of a smooth path is automatically smooth.

Remark 5. This statement is probably well-known, but it is useful and we did not find any proof, so we suggest here one. For our part, we simply realized, during the reading of [Ghy87, Théorème 5.7], where an analog argument is used in the context of Lie groups, that this result depended only on the splitting $E_1 \oplus \cdots \oplus E_n = \mathrm{T}N$.

In [MM20, Lemma 7.2], we use the three-dimensional version of this statement to prove the completeness of a (G,X)-structure (*i.e.* to prove that its developing map is a covering). A little subtlety in the proof of Lemma 4, is that f is not *a priori* assumed to be surjective. The surjectivity of f is maybe not as obvious as in Lemma 1, and is not explicitely justified in [MM20, Lemma 7.2]. Note that this stronger statement (proving the surjectivity of f) is actually not used in [MM20], since the developing map is already known to be surjective (see [MM20, Corollary 7.7], whose proof does not use the Lemma 7.2). Nevertheless, we give here a complete and detailed proof of Lemma 4.

From now on and until the end of this text, we are under the hypotheses and notations of Lemma 4. Note that for any $x \in N$, there exists $\varepsilon > 0$ and an open neighbourhood U of x, such that there are smooth vector fields X_i, $1 \leq i \leq n$, defined on U and generating respectively the distributions E_i on U, such that $\phi(t) \coloneqq \varphi_{X_1}^{t_1} \circ \cdots \circ \varphi_{X_n}^{t_n}(x)$ is well-defined for any $t = (t_1, \ldots, t_n) \in [-\varepsilon\,; \varepsilon]^n$,[1] and $\phi|_{]-\varepsilon;\varepsilon[^n}$ is a diffeomorphism from $]-\varepsilon\,;\varepsilon[^n$ to U.[2]

Fact 6. *f is surjective.*

Proof. For $x \in N$, we consider the set $\mathcal{O}(x)$ of points that can be reached from x by flowing a finite number of times along integral curves of the distributions E_i, and we call $\mathcal{O}(x)$ the *orbit* of x. Note that the set of orbits from a partition of N, *i.e.* that the relation $x \sim y \iff y \in \mathcal{O}(x)$ is an equivalence relation. Now for $x \in N$, the existence of the diffeomorphism ϕ from $]-\varepsilon\,;\varepsilon[^n$ to an open neighbourhood U of x that we described above, shows that $U \subset \mathcal{O}(x)$. Consequently, an orbit is a neighbourhood of each of its points, *i.e.* any orbit is open. Since they form a partition of N, each of them is also closed, and since N is connected, this shows that there is only one orbit in N. But the hypothesis of lifting paths in the direction of the E_i's implies that if $x \in \mathrm{Im}(f)$ and $y \in \mathcal{O}(x)$, then the piecewise curve joining x to y whose parts are integral curves of the E_i's, can be lifted to M, showing that $y \in \mathrm{Im}(f)$. Therefore $\mathrm{Im}(f)$ is saturated by the orbits, and is thus equal to N which is itself an orbit. □

It only remains to lift the paths locally around any point.

Fact 7. *For any $x \in M$, there exists an open neighbourhood U of x, such that for any $\tilde{x} \in f^{-1}(x)$ and for any path γ starting from x and contained in U, γ lifts from $\tilde{x}$.*

Proof. Let $\varepsilon > 0$, U, X_i and ϕ be as described before Fact 6. Let $\gamma\colon [0\,;1] \to N$ be a path starting from x and contained in U, and let $\tilde{x} \in f^{-1}(x)$ ($x \in \mathrm{Im}(f)$ according to Fact 6). Since $\phi|_{]-\varepsilon;\varepsilon[^n}$ is a diffeomorphism from $]-\varepsilon\,;\varepsilon[^n$ to U, there is a continuous map $T\colon [0\,;1] \to\,]-\varepsilon\,;\varepsilon[^n$ such that $\gamma(s) = \phi \circ T(s)$. We introduce the pullbacks $\tilde{X}_i = f^* X_i$. By hypothesis on X_n and ε, the path $s \in [0\,;\varepsilon] \mapsto \varphi_{X_n}^{t_n}(x)$ is well-defined, and by hypothesis on f it lifts from $\tilde{x}$ to a path α, verifying $f \circ \alpha(s) = \varphi_{X_n}^{s}(x)$ and $\alpha(0) = \tilde{x}$. For any $s \in [0\,;\varepsilon]$, we thus have $\mathrm{D}_{\alpha(s)} f(\alpha'(s)) = \frac{d}{ds}\varphi_{X_n}^{s}(x) = Z(f \circ \alpha(s))$ so that $\alpha'(s) = \tilde{Z}(\alpha(s))$. In other words, $\alpha(s)$ is the integral curve of $\tilde{X}_n$ with starting point $\tilde{x}$, and $\varphi_{\tilde{X}_n}^{s}(\tilde{x})$ exists therefore for any $s \in [0\,;\varepsilon]$. The same argument for the path $s \in [0\,;\varepsilon] \mapsto \varphi_{X_n}^{-s}(x)$ shows that $\varphi_{\tilde{X}_n}^{s}(\tilde{x})$ exists for any $s \in [-\varepsilon\,;\varepsilon]$. Then for any $t_n \in [-\varepsilon\,;\varepsilon]$, we repeat this argument for the paths $s \in [0\,;\varepsilon] \mapsto \varphi_{X_{n-1}}^{s} \circ \varphi_{X_n}^{t_n}(x)$ and $s \in [0\,;\varepsilon] \mapsto \varphi_{X_{n-1}}^{-s} \circ \varphi_{X_n}^{t_n}(x)$, whose lifts from $\varphi_{\tilde{X}_n}^{t_n}(\tilde{x})$ show that $\varphi_{\tilde{X}_{n-1}}^{t_{n-1}} \circ \varphi_{\tilde{X}_n}^{t_n}(\tilde{x})$ exists for any $(t_{n-1}, t_n) \in [-\varepsilon\,;\varepsilon]^2$. Repeating this argument, a finite recurrence shows that $\tilde{\phi}(t) \coloneqq \varphi_{\tilde{X}_1}^{t_1} \circ \cdots \circ \varphi_{\tilde{X}_n}^{t_n}(\tilde{x})$ is well-defined for any $t = (t_1, \ldots, t_n) \in\,]-\varepsilon\,;\varepsilon[^n$. By construction $f \circ \tilde{\phi} = \phi$, and $\tilde{\gamma} \coloneqq \tilde{\phi} \circ T$ is thus a lift of γ starting from $\tilde{x}$, which concludes the proof of the fact. □

We now conclude the proof of Lemma 4. Let $\gamma\colon I = [0\,;1] \to N$ be a path starting from $x \in N$ and let $\tilde{x} \in f^{-1}(x)$. For any $t \in I$, since $\gamma(t) \in \mathrm{Im}(f)$ according to Fact 6, there exists according to Fact 7 an open connected neighbourhood U_t of $\gamma(t)$, such that any path starting from $\gamma(t)$ and contained in U_t lifts from any point of $f^{-1}(\gamma(t))$. But γ is continuous, so there exists for any t a connected closed neighbourhood I_t of t such that $\gamma(I_t) \subset U_t$. Since the I_t cover I, a finite number of them cover it, that we can choose to be $[t_i\,;t_{i+1}]$ with $t_1 = 0 \leq \cdots \leq t_n = 1$. Now for any $1 \leq i \leq n$, $\gamma|_{[t_i;t_{i+1}]}$ can be lifted from any point of $f^{-1}(\gamma(t_i))$, which allows us to lift it entirely from $\tilde{x}$ in a finite number of step. This shows that f verifies the path-lifting property, which concludes the proof of Lemma 4 according to Lemma 1. Actually, the following easier argument allows to conclude the proof of Lemma 4 without using Lemma 1.

For $x \in N$, since $x \in \mathrm{Im}(f)$ according to Fact 6, the proof of Fact 7 showed the existence of a connected open neighbourhood U of x and of $\varepsilon > 0$ such that there exists a diffeomorphism

[1]This exists, because the minimal size of the open interval of definition of the local solution to an ODE, is locally bounded from below.

[2]This exists according to the Inverse Mapping Theorem.

$\phi\colon]-\varepsilon\,;\varepsilon[^n \to U$, and such that for any $\tilde{x} \in f^{-1}(x)$ there exists a smooth map $\tilde{\phi}_{\tilde{x}}\colon]-\varepsilon\,;\varepsilon[^n \to M$ verifying $\tilde{\phi}_{\tilde{x}}(0,\dots,0) = \tilde{x}$ and $f \circ \tilde{\phi}_{\tilde{x}} = \phi$. Now, let V denote one of the connected components of $f^{-1}(U)$. The restriction $f|_V\colon V \to U$ still verifies the path-lifting property since V is connected, and it only remains to prove that $f|_V$ is injective to conclude that f is a covering map. There exists $\tilde{x} \in f^{-1}(x) \cap V$, and we denote $\tilde{\phi} = \tilde{\phi}_{\tilde{x}}$. Since $\tilde{\phi}(0,\dots,0) = \tilde{x}$, $f \circ \tilde{\phi} = \phi$, and $]-\varepsilon\,;\varepsilon[^n$ is connected, $\tilde{\phi}(]-\varepsilon\,;\varepsilon[^n) \subset V$. For any $\tilde{y} \in V$, there exists a path $\tilde{\gamma}$ in V joining $\tilde{x} = \tilde{\gamma}(0)$ to $\tilde{y} = \tilde{\gamma}(1)$. Since $\phi\colon]-\varepsilon\,;\varepsilon[^n \to U$ is a diffeomorphism, there exists $T\colon [0\,;1[\to]-\varepsilon\,;\varepsilon[^n$ such that $f \circ \tilde{\gamma} = \phi \circ T$, and since $\tilde{\phi} \circ T$ is a continuous lift of $f \circ \tilde{\gamma}$ starting from $\tilde{x}$, it is equal to $\tilde{\gamma}$, so $\tilde{y} \in \tilde{\phi}(]-\varepsilon\,;\varepsilon[^n)$. Finally $\tilde{\phi}(]-\varepsilon\,;\varepsilon[^n) = V$, and the equality $f \circ \tilde{\phi} = \phi$ implies thus that $f|_V$ is injective, because ϕ is.

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